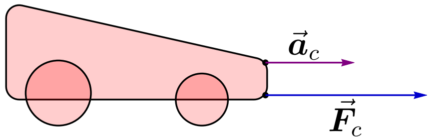
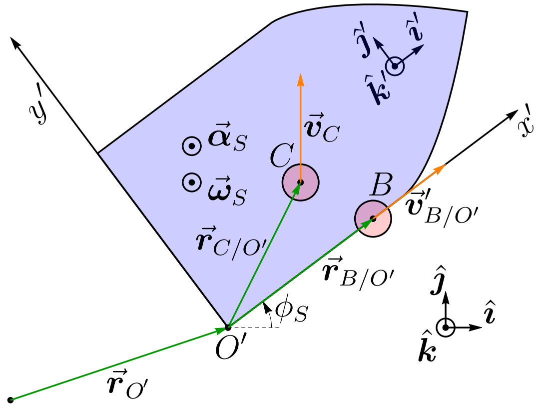
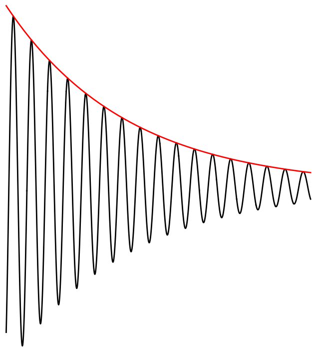
Introductory Dynamics

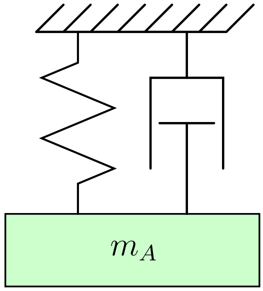
2D Kinematics and Kinetics of Point Masses and Rigid Bodies

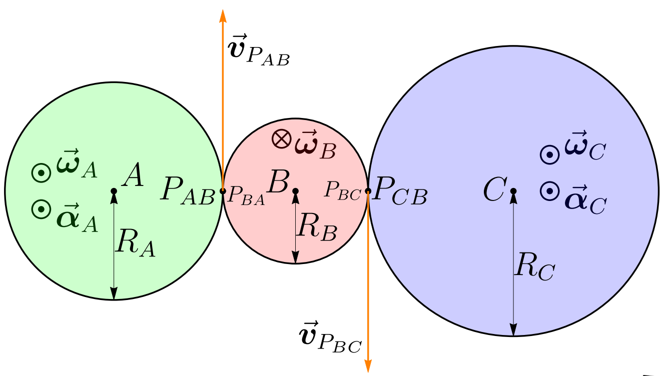
Peter G. Steeneken











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Contents

Part I

Introduction

1 Preface

This textbook provides you with an overview of the methods to analyse and predict the dynamic motion of point masses and rigid bodies in the two-dimensional plane based on Newton's laws. The book is aimed at first-year bachelor students in science and engineering. The recommended prerequisite knowledge for the textbook includes high-school physics, basic calculus (differentiation and integration), vector algebra, linear ordinary differential equations, and knowledge of statics. Furthermore, the gradient is introduced from vector calculus. Most concepts are briefly introduced, such that this textbook might in principle also be used by readers without this prerequisite knowledge.

In case you spot errors, inaccuracies or if you have other suggestions to improve this textbook it would be highly appreciated if you send your feedback to p.g.steeneken@tudelft.nl.

1.1 Guidelines for studying dynamics

This textbook aims to bring you basic understanding of dynamics and provide an engineering toolbox by which you can solve many problems in dynamics, both analytically and numerically. However, it is important to learn how to use this toolbox to apply it in new situations. The only way to learn this well is to solve problems. For this reason, it is very important, after having studied a part of the textbook, to practice solving problems yourself. Always first try to solve the problem yourself before looking at the solution, since you learn much more by finding the solution yourself, even if that costs more time. Getting stuck in solving a problem is probably the experience from which you learn the most. It points out what parts you don't understand well enough, encourages you to study those parts better by reading the book or looking at similar examples, and after solving the problem the experience will ensure you will not easily forget how to solve similar problems. So, don't give up too early or look at the solution, if you persist you often will manage.

Besides teaching you the basics of dynamics, an important aim of this textbook is to teach you to get rid of 'bad habits' you might have acquired in high school by implementing the following guidelines:

1. **Work with equations instead of filling in numerical values while analysing a problem.** The aim is to learn how to solve problems with equations instead of filling in numbers at an early stage. Not only are you more likely to make errors when filling in numbers, it is also much harder to detect those errors afterwards (e.g. by checking units). Moreover, when having an equation as solution, it has a larger degree of validity and can be used to predict trends and optimise designs. Only fill in numbers after having reached the final equation.
2. **Work with vectors instead of scalars.** A position, force, moment, velocity or acceleration in 2 or 3 dimensional space is always a vector. *Therefore you should initially always use vector notation in 2D and 3D*, and only work with scalar values when appropriate, e.g. after having projected the vectors on a coordinate axis. Correct vector usage and notation will become increasingly important when you progress with your studies, so best get used to it early. Note that most problems in this book will focus on motion in the 2-dimensional xy-plane. Nevertheless, even in that case moment and angular velocity vectors point along the z direction. Moreover, most vector equations that are presented in this textbook will still be valid in 3D, unless they have a subscript $_{2D}$.
3. **Using a structured approach instead of jumping to conclusions too soon.** *In the analysis of dynamic systems, we will not assume anything, but use structured procedures to solve problems.* For example, if a mass moves over a surface, we will not jump to the conclusion that the upward normal force equals the weight of the mass (which is not always the case for a curved surface), but determine the normal force from Newton's laws.
4. **Solve problems based on understanding, instead of by memorising example problems, and copying the methodology without understanding how it works.** Although this memorising and solving many standard problems might seem effective for passing exams, it will leave you helpless when you encounter a problem that you did not see before. *Therefore, focus on building understanding and ask questions if things are not clear.*

This textbook aims to form a bridge between high-school physics and academic literature, providing more rigorousness and insight into the material. Available textbooks are often either too advanced and complex, or do not offer sufficient depth, insight or rigorousness in my view. This textbook provides a solid basis in dynamics of which I hope that it will help you to bridge the gap towards understanding more advanced academic books and articles.

1.2 Topics

The outline of the topics that will be discussed in this textbook is as follows:

1. Introduction, elemental concepts, forces, vectors and mathematics
2. Dynamics of point masses
   a) Kinematics
   b) Kinetics
   c) Work and energy
   d) Impulse and momentum
3. Dynamics of rigid bodies
   a) Kinematics
   b) Kinetics
   c) Work and energy
   d) Angular impulse and momentum
4. Vibrations
5. Solution strategy dynamics

Although all equations and theory will be derived and presented for motion in 3D space (unless specifically indicated otherwise for special cases), most examples and problems will focus on point masses and rigid bodies moving in a 2D plane. The dynamics of deformable objects will not be part of the textbook. Several section headings are marked with a * at the end of the section title, to indicate that the main text can be followed without reading these parts because they contain derivations or background information. The following introductory chapters might be skipped if they are already familiar. The core of the textbook starts in Ch. 5.

1.3 Text structure

In this textbook, the techniques and models that scientists and engineers need to design and analyse dynamical systems will be introduced step by step, starting from Newton's laws. Eventually at the end of the textbook we will have discussed a toolbox of equations and techniques that can be valuable for your future career.

The text consists of different parts:

1. Descriptions and explanations
2. Derivations of key equations and concepts

3. Concepts and techniques in dynamics
4. Practical examples, and sample problems that illustrate the use of these techniques

Although derivations of equations might be skipped without following the main line of the textbook, it is highly recommended to read them, since they do provide more insight into the theory and demonstrate beautifully how Newton's laws, despite their simplicity, can account for a tremendous set of complex phenomena in dynamics.

To highlight important concepts, they are introduced like this:

Concept. *Principle of linear impulse and momentum*

$$\sum_j m_j \vec{\boldsymbol{v}}_j\left(t_1\right)+\sum_{i,j} \int_{t_1}^{t_2} \vec{\boldsymbol{F}}_{i,j} \mathrm{d} t=\sum_j m_j \vec{\boldsymbol{v}}_j\left(t_2\right) \tag{1.1}$$

Similarly, derivations are marked like this:

Derivation. *We will here derive the principle of impulse and momentum from Newton's laws.*

Examples to illustrate how to apply the concepts and techniques are provided in an example environment, with the letter in front of the example indicating the difficulty level (S: simple, M: medium and A: advanced).

S **Example 1.1** We consider the following example situation.

Exemplary solution
The dynamics of this system can be solved as follows.

Problems are included in these notes as follows:

M **Problem 1.1** This is a test problem.

1.4 Acknowledgements

In 2021 I started writing this textbook to support the course Introductory Dynamics that I teach for first year BSc. students, motivated by Heike Vallery who wrote a book on Advanced Dynamics [9] and also provided me with a template [9] and contributed parts of her notes that have been reused in adapted form in Ch. 2 and Ch. 3. I would like to thank Jaap Meijaard for thorough review of the full text. I also would like to thank Heike Vallery, Arno Stienen, Niels Bouman and Boris Ullrich, for their help and feedback on the manuscript. I also acknowledge the support from the department of Precision and Microsystems Engineering and the Mechanical Engineering faculty of TU Delft in writing this textbook. TU Delft Open Textbooks has helped me to publish the book. I provide it with CC BY licence, with the Latex source code and images on Zenodo (see DOI on page 2), in order to facilitate distribution and education, and to enable others to reuse it and expand on it, which is encouraged and permitted as long as a reference to this textbook is provided. Note that none of the underlying dynamics theory presented in this textbook is new. All of it is based on beautiful scientific work that was developed over several centuries by many scientists and educators, most notably Newton [7, 6] and Euler [3].

2 Notation, Math, and Engineering Basics

This chapter will discuss notation conventions, unit usage, significant digits and repeat mathematical background knowledge that will be needed to follow this textbook.

2.1 Notation

In this section we discuss the used notation, labelling and typesetting in this textbook. Sec. 2.1 and Sec. 2.2 are an adaptation of the text of Vallery and Schwab [9].

2.1.1 Typesetting of scalars, vectors, and matrices

There are several ways to typeset scalars, vectors, unit vectors, and matrices as depicted in Table 2.1. In particular for vectors and unit vectors there are multiple methods applied. Although in many works vectors are only written with boldface italics, it is hard to clearly write a boldface letter in handwriting. Therefore we here use a combination of boldface and vector arrow above the letter for clarity and similarity to the handwritten vector. Similarly, for unit vectors (vectors with a magnitude of 1), we use a boldface letter with a hat $\hat{}$ above it. All variables are italics, also if they appear in subscripts or superscripts. Matrices are non-italic and also names and texts in sub or superscripts are non-italic except if they are single letters[8].

Table 2.1: Notation of scalars, vectors, unit vectors and matrices

	This book	Other notations
Scalars	F	
Vectors	$\vec{\boldsymbol{F}}$	$\vec{F}$, $\underline{F}$, $\boldsymbol{F}$, $\mathbf{F}$
Unit vectors	$\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\rho}}$	$\mathbf{u}_x$, $\mathbf{e}_1$, $\mathbf{i}$, $\hat{\boldsymbol{x}}$
Matrices	$\mathbf{R}$	$\mathcal{R}$

2.1.2 Quantities, labels and subscripts

In this textbook we generally follow the guidelines set out in the Red Book [2]. To label objects or points we either use letters like $A, B, \ldots$, numbers $i = 1, 2, \ldots$ or short words like rope or ball. For adding extra information and distinguishing quantities, we can add subscripts like in m_{rope} and m_{ball}. Multiple labels can be separated in the subscript by commas. Italic integer variables, like i in F_i (with $i = 1, 2, \ldots$) can indicate that multiple quantities $F_1, F_2, \ldots$ exist, each with a different value of i. Although labels are not always essential, make sure to always use sufficient labels to uniquely identify each quantity. So, if there is only one mass in your problem, it is fine to just m to identify it. However, if there are two point-masses, subscripts m_A and m_B are needed for unique identification. With an axis coordinate as a subscript, like x in F_x, we indicate a projection of a vector $\vec{\boldsymbol{F}}$ along the x-axis.

A special notation for relative position, velocity and acceleration vectors is used (see also Sec. 3.1.2). A position vector that points to a point A from a point P will be denoted as $\vec{\boldsymbol{r}}_{A/P} = \vec{\boldsymbol{r}}_A - \vec{\boldsymbol{r}}_{/P}$, which reads: "position of A *with respect to* P". Analogous notation will be used for other relative quantities, such as angles, velocities and accelerations (see Sec. 3.1).

For components of 3-dimensional vectors, there are three possible ways of notation of which we will usually use the last one because it is the shortest to write:

$$\vec{\boldsymbol{r}} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} r_x & r_y & r_z \end{pmatrix}^{\mathrm{T}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}}. \tag{2.1}$$

The superscript $^{\mathrm{T}}$ indicates the transpose of the vector (or matrix), and is sometimes used to convert the column vector to a row vector to save space. For components of m-dimensional vectors, one can also use the *index notation*, with unit vectors written like $\hat{\boldsymbol{e}}_1 = \hat{\boldsymbol{\imath}}$, which has the advantage that it can be written even more compactly as a sum:

$$\vec{\boldsymbol{r}} = \begin{pmatrix} r_1 \\ r_2 \\ \ldots \\ r_m \end{pmatrix} = \begin{pmatrix} r_1 & r_2 & \ldots & r_m \end{pmatrix}^{\mathrm{T}} = \sum_{i=1}^{m} r_i\hat{\boldsymbol{e}}_i. \tag{2.2}$$

Elements of $m \times n$-dimensional matrices receive two indices, for row and column:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}. \tag{2.3}$$

Note that in dynamics, we use Newton's notation (also called dot notation or fluxions) for time derivatives, indicating them by a dot above the variable:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{2.4}$$

$$\ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} \tag{2.5}$$

In mathematics the prime symbol $'$ is often used for derivatives like $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$. In dynamics it is advisable not to use the prime symbol to avoid confusion, and instead clearly indicate with respect to what variable the derivative is taken e.g. $\frac{\mathrm{d}f}{\mathrm{d}x}$ or $\frac{\mathrm{d}f}{\mathrm{d}s}$. Also clearly indicate if it is a total derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$ or partial derivative $\frac{\partial f}{\partial x}$. In this textbook the prime symbol $'$ will not be used for derivatives but is sometimes used as a label. Finally, when making multiplications of scalar quantities, both $\cdot$ and $\times$ are used, such that $a \cdot b = a \times b$. For vectors these symbols indicate the dot and cross product which clearly are different: $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}} \neq \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}$.

2.2 Working with units

There are several rules when working with units that are not always obvious, and sometimes not correctly applied, even in standard textbooks. Errors with units have led to serious confusions and accidents like the collapse of a Mars orbiter and an aeroplane running out of fuel in mid-air. It is therefore important for every engineer to be able to correctly work with units and we repeat the most important rules here.

In mechanics, we often deal with scalar variables that represent physical quantities that have a measurement unit, like forces, velocities, or positions. Scalar variables consist of the *product* of two elements: a number and a measurement unit.

S **Example 2.1** Force can be expressed as the product of a numerical value and the measurement unit newton: $F = 3 \cdot \mathrm{N} = 3\,\mathrm{N}$.

Following the ISO norm [5] and the “Red Book” [2], for quantities that have a measurement unit, the numerical value of a scalar variable Q that represents such quantities is denoted by $\{Q\}$, and the measurement unit is denoted by $[Q]$, such that

$$Q = \{Q\} \cdot [Q]. \tag{2.6}$$

S **Example 2.2** A mass $m = 3.5\,\mathrm{kg}$ has the numerical value $\{m\} = 3.5$ and the unit $[m] = \mathrm{kg}$. The same quantity value can be expressed as $3500\,\mathrm{g}$. Then, the numerical value is $\{m\} = 3500$, and $[m] = \mathrm{g}$.

Note that square brackets only have meaning when placed around the physical quantity. For axes labels one commonly uses round brackets [4, 8] around the unit like for a time axis: t (s). Another option [1] is to use division: t/s. Generally, the letters of units are printed in roman (upright), those of scalar variables in italic.

S **Example 2.3** For time t,
good use is: $t = 4\,\mathrm{s}$, $t = 4\,(\mathrm{s})$, $t/\mathrm{s} = 4$, or $[t] = \mathrm{s}$,
not good use is: ~~$t = 4\,[\mathrm{s}]$~~, or ~~$t\,[\mathrm{s}]$~~. Besides paying attention to the units, also you need to make sure that if you have a vector on the left side of the = sign, you also need to have a vector on the right side.

From Eq. (2.6) it can be seen that every scalar equation in mechanics can be split into two equations, one for the numerical values, and one for the units. For example, an equation like $F = ma$, with $F = 1\,\mathrm{N}$, $m = 1\,\mathrm{kg}$ and $a = 1\,\mathrm{m/s^2}$ can be split into:

$$F = m \cdot a \tag{2.7}$$

$$\{F\}[F] = \{m\}[m] \cdot \{a\}[a] \tag{2.8}$$

$$\{F\} = \{m\} \cdot \{a\} \rightarrow 1 = 1 \cdot 1 \tag{2.9}$$

$$[F] = [m] \cdot [a] \rightarrow \mathrm{N} = \mathrm{kg} \cdot \mathrm{m/s^2} \tag{2.10}$$

When making a calculation it is important that in the end result both the numerical values and the units are correct since obviously, in equations, the units on both sides of the equal sign need to be identical, otherwise the equality can never hold. Normally, the units automatically work out well if you work with SI units. In some cases, one has to deal with equations like $a = c_1 t^2$. In that case the unit of the constant c_1 is not given, but can be obtained from the equation $[a] = [c_1][t^2]$, which can be written as $\mathrm{m/s^2} = [c_1] \cdot \mathrm{s^2}$, from which it follows that the unit of c_1 is: $[c_1] = \mathrm{m/s^4}$.

A general advice is to work as long as possible with variables, and only to substitute numbers and units in the final solution. Units almost always help as a part of plausibility checks for calculations. Moreover, writing the numerical value and unit in each step of the derivation is much more work, and errors in numerical values are harder to find than in equations.

In this textbook we mostly work with the International System of Units, also called SI units, the SI system, or metric units, which are the world's most widely used system of measurement. If you have to work with non-SI units, as a habit first convert it to the SI unit as a first step to prevent problems later on. Often this conversion is just a matter of multiplying the non-SI unit value by a constant factor, e.g. $1.0\,\text{inch} = 1.0\,\text{inch} \times 0.0254\,\text{m/inch}$, however in some cases the conversion is more complex, like when converting temperature in Fahrenheit to degrees Kelvin. Preferably also work in the base SI unit, remove any factors related to prefixes (like converting mm to m by dividing by a factor 1000) before starting calculations, using scientific notation (e.g. 3.02×10^8) for very large or small values.

There are 7 base SI units, and the only base SI units relevant in this textbook are m, s and kg. There are 22 derived SI units of which we use 6 in this textbook: the $\text{rad} = \text{m/m}, \text{Hz} = 1/\text{s}, \text{N} = \text{kg m/s}^2, \text{Pa} = \text{N/m}^2, \text{J} = \text{Nm}$, and $\text{W} = \text{J/s}$. The unit to use for a certain quantity or parameter can often be determined by evaluating known dynamics equations. As an example, since x has unit m and t has unit s, we can use the equation for acceleration $a = \frac{\text{d}^2 x}{\text{d}t^2}$ to determine that acceleration a has the unit m/s^2 (note how the square sign ^{2} is placed behind the t but before the x in the second derivative notation). Similarly, one obtains from $F = m \times a$ that $1\,\text{N} = 1\,\text{kg} \times 1\,\text{m/s}^2 = 1\,\text{kg m/s}^2$. This method can often help to find the right unit by using a known equation, such that units don't have to be memorised. Keep in mind that the equations in this textbook only work correctly if one uses SI units or units derived from SI units (like N or W).

In some cases the scalar variable represents a unitless quantity. Examples of unitless quantities are integer numbers (for example the number of wheels under a car) or ratios of quantities with the same unit (for example the ratio of two distances). Most mathematical functions, like trigonometric or logarithmic functions, only accept unitless quantities. For example there is no mathematical way to calculate $\tan(1\,\text{m})$. So, always make sure only unitless numbers or ratios are used. E.g. $\log(x_2/x_1)$ is fine, but $\log(x_2)$ cannot be calculated if $[x_2] = \text{m}$. It is important to note that additions and subtractions can only be performed between scalar quantities of the same unit, while multiplications, divisions and powers can be taken on quantities of different units as long as the operation is also applied to the respective units.

A special case are angles, even though they usually have degrees $^\circ$ or rad behind them, these angle designations are actually unitless numbers, which measure e.g. the ratio of the length of a circular arc divided by the length of the perimeter of a full circle. The degree or radian symbol only indicates how the trigonometric function should be applied. When not otherwise specified in this textbook we work with radians. Since angles are unitless, most mathematical functions can accept angles as an argument.

When dealing with vectors, one also needs to specify the direction of the vector using one or more unit vectors, so for specifying a vector one uses the product of a number, a measurement unit and at least one unit vector. Subtraction and addition is only possible for vectors with the same unit, scalar products are possible as long as the units are also multiplied, so for instance $\vec{\boldsymbol{v}} \times t = \Delta\vec{\boldsymbol{x}}$ is a correct equation, because m/s $\times$ s $=$ m and because quantities on both sides of the $=$ sign are vectors.

Expressions and equations may contain mixtures of quantities given with their numerical value and quantities given as variables. In all cases one needs to ensure that the units on both sides of the equal sign $=$ are equal, and that one cannot equate a scalar to a vector or matrix. If a vector quantity evaluates to zero one we use the zero vector $\vec{\mathbf{0}}$, which is a vector with magnitude zero and no direction.

S **Example 2.4** For a particle of mass $m = 3\,\text{kg}$ having acceleration vector $\vec{\boldsymbol{a}} = 1\,\text{m/s}^2\hat{\boldsymbol{\imath}}$, the resultant force vector $\vec{\boldsymbol{F}}$ on the particle can be calculated as:
Correct is: $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} = 3\,\text{kg} \times 1\,\text{m/s}^2\hat{\boldsymbol{\imath}} = 3\,\text{kg m/s}^2\hat{\boldsymbol{\imath}} = 3\,\text{N}\hat{\boldsymbol{\imath}}$.
*In*correct is: ~~$\vec{\boldsymbol{F}} = 3 \cdot \vec{\boldsymbol{a}}\,\text{N}$~~, because the vector $\vec{\boldsymbol{a}}$ has unit m/s^2 such that the units on both side of the equal sign are different.

S **Problem 2.1** A particle performs a harmonic oscillation, such that its location coordinate x is a function of amplitude x_0, angular frequency Ω, and time t:

$$x = x_0 \sin(\Omega t). \tag{2.11}$$

Fill in suitable units for x, x_0, Ω, and t and check the equation for consistency.

S **Example 2.5** Which of these expressions make correct use of units, for mass m, time t, force F, angular velocity ω, and angle θ? *Correct the mistakes.*

A. $m = 20$
B. $m = 3\,\text{N}$
C. $F = 3 \cdot m\,[\text{N}]$
D. $\text{F} = 20N$
E. $[m] = \text{kg}$
F. $m/[m] = 20$
G. $t = 3\,\text{s}$
H. $\omega = 2\,\text{rad/s}$
I. $\theta = \omega \cdot t = 2 \cdot t\,[\text{s}]$
J. $\{m\} = 20$
K. $\omega = t^2$
L. $\omega = 1\,\text{rad/s} \cdot t^2$
M. $F(t) = \cos(1\,\text{rad/s} \cdot t)\ \text{N}$
N. $F(t) = 3e^{-t}\ \text{N}$
O. $F(t) = \ln(t^2/\text{s}^2)\ \text{kg} \cdot \text{m/s}^2$

Exemplary solution

$m \neq 20$	unit omitted; $m = 20\,\mathrm{kg}$
$m \neq 3\,\mathrm{N}$	wrong unit for mass quantity; $m = 3\,\mathrm{kg}$
$F \neq 3 \cdot m\,[\mathrm{N}]$	unit omitted and unit inserted with wrongly used square brackets; $F = 3\,\mathrm{m/s^2} \cdot m$
$\mathrm{F} \neq 20N$	roman variable, italic unit symbol; $F = 20\,\mathrm{N}$
$[m] = \mathrm{kg}$	correct!
$m/[m] = 20$	correct!
$t = 3\,\mathrm{s}$	correct!
$\omega = 2\,\mathrm{rad/s}$	correct!
$\theta = \omega \cdot t \neq 2 \cdot t\,[\mathrm{s}]$	wrong unit and [s] is wrong; $\theta = 2\,\mathrm{rad/s} \cdot t$
$\{m\} = 20$	correct!
$\omega \neq t^2$	unit mismatch; e.g. $\omega = 1\,\mathrm{rad/s^3} \cdot t^2$
$\omega \neq 1\,\mathrm{rad/s} \cdot t^2$	unit mismatch; e.g. $\omega = 1\,\mathrm{rad/s^3} \cdot t^2$
$F(t) = \cos(1\,\mathrm{rad/s} \cdot t)\ \ \mathrm{N}$	correct !
$F(t) \neq 3e^{-t}\ \mathrm{N}$	exponent should not have units; e.g. $F(t) = e^{-t/\mathrm{s}}\ \ \mathrm{N}$
$F(t) = \ln(t^2/\mathrm{s}^2)\ \mathrm{kg \cdot m/s^2}$	correct !

2.3 Precision, significant digits and errors

When designing new systems, it is essential to be sure that they satisfy the required specifications. Therefore, when working with numerical values, it is essential to be aware of the precision of these numbers. Neglecting the consideration of the precision can lead to big problems, like parts of machines that do not fit together. Uncertainties in quantities with respect to the average value can occur both due to actual variations (in time) in the quantity that is measured and due to uncertainties in the measurement technique that is used to determine the quantity.

2.3.1 Precision and significant digits

The most common method to determine the uncertainty in a quantity is to measure the quantity q a large number of times and determine the average value q_{avg} and standard deviation σ_q. For example we measure the diameter of a 1 euro cent coin N times with a calliper, analyse the results and obtain $d = d_{\mathrm{avg}} \pm \sigma_d = (16.20 \pm 0.05)$ mm. The standard deviation σ_d is a measure of the uncertainty (imprecision) in a single measurement[1] of the value of the

[1] Note that the uncertainty in the average value of all the N measurements is lower than the uncertainty in a single measurement: $\sigma_{d_{\mathrm{avg}}} \approx \sigma_d/\sqrt{N}$. So, taking more measurements of

coin diameter d. If many different euro cent coins are measured variations between the euro cent coins can also increase the uncertainty, thus both measurement uncertainty and object variations can contribute to the total uncertainty. Assuming a Gaussian distribution of the measurement values, the value of the standard deviation gives a probability of 68% that the diameter of the coin is between 16.15 mm and 16.25 mm.

Significant digits (also called significant figures) are an approximate way to indicate these kinds of uncertainties in a value. Since adding or removing one significant digit changes the precision of the value by a factor 10, it is a quite rough indicator of the precision. Nevertheless, even this rough estimate can protect you from making grave mistakes. For instance, you buy a piece of wood with a width of $w = 100\,\text{cm}$ to fix a door in your house. When installing it you notice that it is only 99.6 cm such that you have an ugly gap of 0.4 cm (or you might find the door doesn't close because $w = 100.4$ cm). If you go back to the salesman you might not get your money back, since 100 cm means that the width can have any value between 99.5 cm and 100.5 cm, instead you should have bought a piece of wood with a width of $w = 100.0$ cm.

In the example of the euro cent coin, the diameter of the coin would be indicated as $d = 16.2$ mm. The last digit indicates the uncertainty, showing that with reasonable certainty the value will be between 16.3 mm and 16.1 mm. Alternatively you can continue working with the full expression (16.20 ± 0.05) mm, and use error analysis to determine the error and precision. These methods also can be used to determine the error propagation, for instance in case a quantity q is determined from d by a function $q = f(d)$, an error in d will also result in an error in q. If we determine the area of the euro cent coin using the equation $A(d) = f(d) = \frac{1}{4}\pi d^2$ we find, using the Taylor expansion for small errors σ_d, that $A = f(d_{\text{avg}} \pm \sigma_d) \approx f(d_{\text{avg}}) \pm \sigma_d \frac{\partial f}{\partial d} = \frac{1}{4}\pi(d_{\text{avg}}^2 \pm 2d_{\text{avg}}\sigma_d)$, such that the error (standard deviation) in A can be determined by: $\sigma_A \approx \sigma_d \frac{\partial f}{\partial d}$. However, although very important, further discussion of these methods is out of the scope of this textbook.

When reporting measured quantities in engineering, the number of significant digits should always be estimated and reported[2]. In case estimating precision is difficult, a conservative estimate of the number of significant digits

the same quantity is a way to reduce the uncertainty.

[2]Besides the precision, that indicates the variation of the quantity around the average value, there can also be a deviation of the average measured value with respect to the actual value. For instance the calliper might have an offset, such that it always measures 0.05 mm less than the actual value. that would be measured by a more accurate instrument. This is called a systematic error, and its magnitude is related to the *accuracy* of the quantity.

is recommended. Significant digits can sometimes only be properly indicated using scientific notation, which is notation of a number by a single digit before the decimal point and an appropriate exponent. For example, if there are two significant digits, a mass should be written as 1.2×10^2 kg and not as 120 kg (which indicates three significant digits). The most important point to note is that trailing zeros, like in 1.000, right of the last non-zero digit are all significant, while the leading zeros like in 0.001, left of the first non-zero digit, are not significant. Some examples:

- 1.2×10^2 kg $\neq$ 120 kg
- 2.000 kg = 2000 g = 2.000×10^3 g
- $1.8{\times}10^{-3}$ m = 0.0018 m = 1.8 mm

In the first case the quantity on the left has two significant digits, while the quantity on the right has three significant digits, because the zero is a trailing zero, just like the zeros in the second case. In the third case, the zeros in 0.0018 m are not significant, because they are leading zeros, such that there are only two significant digits.

2.3.2 Making calculations with significant digits

When making calculations based on quantities with significant digits, the number of significant digits can change. If the number of significant digits reduces, the value is calculated and rounded to the nearest number with the determined number of significant digits. The following rules are most important to determine the number of significant digits of a calculated value:

- There is no uncertainty in countable, integer numbers, or mathematical quantities like π. They should be treated as having an infinite number of significant digits.
- To facilitate working with significant digits, write the number in scientific notation to remove all zeros on the left, but keep all trailing zeros on the right of the value: $d = 0.0162$ m= 1.62×10^{-2} m. Then count the number of significant digits, in this case 3.
- In multiplication or divisions, the resulting number of significant digits is equal to the figure with the fewest significant digits. E.g. in $F = m \cdot a = 1.0\,\mathrm{kg} \times 3.124\,\mathrm{m/s^2} = 3.1\,\mathrm{N}$, the resulting number of significant digits is two, equal to that of the mass, which has the lowest number of significant digits.
- In additions or subtractions, the last significant digit of the calculated result should be equal to the last (smallest) significant digit of the least

significant quantity from which it is determined. E.g. $\Delta x = x_2 - x_1 = 2.156\,\mathrm{m} - 2.13\,\mathrm{m} = 0.03\,\mathrm{m}$. The result of the calculation is 0.026 m, but the last significant digit in 2.13 m (the 3) is at two positions right of the decimal point, such that only 1 significant digit remains in the final rounded result.

- When applying functions to quantities, like $\tan\frac{x_1}{x_2} = \tan(0.988) = 1.52$, the number of significant digits is kept the same as the argument. Note that his rule, although often sufficient, is not always completely accurate, e.g. when working with power functions. More advanced methods are outside the scope of this textbook.
- Only determine significant digits at the end of a calculation. For intermediate results in the calculation one should keep a sufficiently large number of digits to prevent additional imprecision due to rounding.

S **Example 2.6** Which of these expressions make correct use of significant digits and units, for mass m, time t, force F, angular velocity ω, and angle θ?

A. $F = 0.0020\,\mathrm{kg} \cdot 300\,\mathrm{m/s^2} = 0.60\,\mathrm{N}$
B. $t = 0.10\,\mathrm{s} - 1.0\,\mathrm{ms} = 99\,\mathrm{ms}$
C. $\theta = \pi \times 10^{-3.0\,\mathrm{ms}/(1.0\,\mathrm{ms})}\mathrm{rad} = 3.1 \times 10^{-3}\,\mathrm{rad}$

Exemplary solution

$F = 0.60\,\mathrm{N}$	correct! 2 significant digits are kept in multiplication.
$t \neq 99\,\mathrm{ms}$	$t = 0.10\,\mathrm{s}$. Most significant figure in first value is 10 ms, same should hold for final result after subtraction.
$\theta = 3.1 \times 10^{-3}\,\mathrm{rad}$	correct! $10^{-3.0} \approx 1.0 \times 10^{-3}$.

2.4 Solving first-order differential equations

Several mathematical techniques are important in dynamics, and for that reason we dedicate space to repeat them in this textbook. Here we discuss solving differential equations and in the next section we discuss complex numbers. In the next chapter we discuss vectors and linear algebra methods.

In kinematics we will often encounter *ordinary differential equations* (ODEs) that need to be solved in order to analyse the motion of objects. Solving an ODE always involves finding an unknown function $y(x)$ based on a known ODE. The function $y(x)$ has a dependent variable y like the position s, speed v or acceleration a of a point, and an independent variable x, which can be the time t, position s or velocity v of the point. Realise that while in mathematics courses you usually only deal with variables y and x, in dynamics different variables are used, like s, v, a and t.

Although we will deal with second order ODEs in Ch. 13, in most of this textbook we will deal with first-order ODEs that contain a known function $f(x)$ (for instance a known position $s(t)$) and the unknown function $y(x)$ that we want to determine. The ODE can occur in either of these forms:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = y(x) \quad (2.12)$$

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = f(x) \quad (2.13)$$

The convenient aspect of these first-order differential equations is that they can be solved using differentiation and integration. In case of an ODE of the form of Eq. (2.12), the solution can be found by taking the derivative of the known function $f(x)$ to find the solution of the ODE $y(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$. For the second case shown in Eq. (2.13) solving the ODE requires integration, and can either be carried out by using the indefinite integral, or by using the definite integral. We will outline both integration procedures. In the following three subsections we discuss the basic differentiation and integration techniques in some more detail.

2.4.1 Differentiation by chain and product rule

When dealing with an explicit function of time $v_s(t)$ taking the derivative is straightforward, e.g. $a_s(t) = \frac{\mathrm{d}v_s(t)}{\mathrm{d}t}$. However, when dealing with functions that are implicit functions of time like $v_s(s)$, where the time dependence of the position s is given by $s(t)$, the chain rule is needed. In that case we use the chain rule like this:

$$a_s(t) = \frac{\mathrm{d}v_s(s(t))}{\mathrm{d}t} = \frac{\mathrm{d}v_s(s)}{\mathrm{d}s}\frac{\mathrm{d}s(t)}{\mathrm{d}t} \tag{2.14}$$

Sometimes the product rule for differentiation is also needed:

$$\frac{\mathrm{d}F(t)\cdot v(t)}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}t}\cdot v(t) + F(t)\cdot\frac{\mathrm{d}v}{\mathrm{d}t} \tag{2.15}$$

2.4.2 Indefinite integral

In case of Eq. (2.13) the ODE can be solved by integration using the indefinite integral to obtain $y(x)$ as follows:

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = f(x) \tag{2.16}$$

$$\mathrm{d}y = f(x)\mathrm{d}x \tag{2.17}$$

$$\int \mathrm{d}y = \int f(x)\mathrm{d}x \tag{2.18}$$

$$y(x) = F(x) + c_1 \tag{2.19}$$

Here, $F(x)$ is a primitive function of $f(x)$ such that $\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$ and c_1 is an unknown integration constant. This constant can be determined if the value of y is known for one value of $x = x_0$. E.g. if $y(x_0) = y_0$, then we have $y_0 = F(x_0) + c_1$, from which we find the value of the integration constant $c_1 = y_0 - F(x_0)$ and by substituting this value of c_1 in Eq. (2.19) the full solution of the ODE is found to be $y(x) = [F(x) - F(x_0)] + y_0$.

2.4.3 Definite integral

Besides taking the indefinite integral, one can also choose to use a definite integral to solve the ODE, which is often a bit shorter, since the known condition $y(x_0) = y_0$ can be directly taken as one of the boundaries of the definite integral. The definite integral is obtained as follows:

$$\begin{aligned}
\frac{\mathrm{d}y(x)}{\mathrm{d}x} &= f(x) && (2.20)\\
\mathrm{d}y &= f(x)\mathrm{d}x && (2.21)\\
\int_{y(x_0)}^{y(x)} \mathrm{d}y_i &= \int_{x_0}^{x} f(x_i)\mathrm{d}x_i && (2.22)\\
[y_i]_{y_0}^{y} &= [F(x_i)]_{x_0}^{x} && (2.23)\\
y - y_0 &= F(x) - F(x_0) && (2.24)\\
y(x) &= [F(x) - F(x_0)] + y_0 && (2.25)
\end{aligned}$$

Note that since the definite integral evaluates the difference $F(x) - F(x_0)$, any integration constant c_1, that appeared in the indefinite integral Eq. (2.19), is automatically eliminated and does not need to be determined. Eventually the result of both methods, indefinite or definite integration, is of course identical. When taking a definite integral, it is essential to ensure that the boundaries of the integrals on the left and right side of the equation are corresponding to identical points of the curve $y(x)$: $y_0 = y(x_0)$ and $y = y(x)$. Note that we have used the variable x as the upper boundary of the definite integral, which allows us to find the function $y(x)$ for any value of x. Note also that for clarity when taking the integral on both sides we have replaced the variables x and y inside the integral by the variables x_i and y_i (the result of the integral is of course independent of the symbol we choose to integrate over), to clearly distinguish them from the integration boundaries x and y. We take the integral of the function $f(x_i)$ over the variable x_i, that runs from x_0 up to x. Or, in other words, with the definite integral we determine the area under the $f(x_i)$ graph, for values of x_i running from $x_i = x_0$ to $x_i = x$. If the integral runs from $x_i = x_1$ to $x_i = x_2$, Eq. (2.22) can also be written in the following form, that we will use regularly:

$$y(x_2) = y(x_1) + \int_{x_1}^{x_2} f(x_i)\mathrm{d}x_i \tag{2.26}$$

From this equation it is clear that if the value of y is known at one value of x_1, then $y(x_1)$ can be determined such that evaluation of the integral enables us to determine $y(x_2)$ at any other value x_2.

2.5 Complex numbers

Working with complex numbers can simplify the analysis of the dynamics in many cases. In particular when working with vibrations, we will use complex numbers a lot to simplify solving differential equations, see Ch. 13. When you know how to work with complex numbers, it saves you remembering many trigonometric equations. Here we will only summarise the most useful relations for complex numbers that are part of most introductory mathematical analysis courses.

A complex number y can be written as the sum of a real part y_r and an imaginary part y_i multiplied by the imaginary unit i. We now summarise some of the properties of complex numbers that you should be able to use. Especially the first three are important, since by using the normal rules for calculating with scalars the rest of the properties can be proven from them.

$$i^2 = -1 = e^{i\pi} \tag{2.27}$$

$$y = y_\mathrm{r} + iy_\mathrm{i} = |y|e^{i\varphi} \tag{2.28}$$

$$e^{ix} = \cos x + i \sin x \quad \text{Euler's formula} \tag{2.29}$$

$$\Re y = \mathrm{Re}(y) = y_\mathrm{r} = |y| \cos\varphi \quad \text{Real part of } y \tag{2.30}$$

$$\Im y = \mathrm{Im}(y) = y_\mathrm{i} = |y| \sin\varphi \quad \text{Imaginary part of } y$$

$$\varphi = \arctan \frac{y_\mathrm{i}}{y_\mathrm{r}} \quad \text{Argument of } y \text{ for}^3\ y_\mathrm{r} > 0 \tag{2.31}$$

$$y^* = \overline{y} = y_\mathrm{r} - iy_\mathrm{i} \quad \text{Complex conjugate (c.c.)} \tag{2.32}$$

$$[f(y_\mathrm{r} + iy_\mathrm{i})]^* = f(y_\mathrm{r} - iy_\mathrm{i}) \quad \text{Replace } i \text{ by } -i \text{ for c.c.}^4 \tag{2.33}$$

$$|y| = \sqrt{y_\mathrm{r}^2 + y_\mathrm{i}^2} = \sqrt{y \cdot y^*} \quad \text{Absolute value} \tag{2.34}$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \tag{2.35}$$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \tag{2.36}$$

$$\frac{1}{y} = \frac{1}{y}\frac{y^*}{y^*} = \frac{y^*}{|y|^2} \tag{2.37}$$

$$y^x = |y|^x e^{ix(\varphi + 2n\pi)} \quad , n = \ldots, -1, 0, 1, 2, \ldots \tag{2.38}$$

S **Example 2.7** If you can work well with complex numbers, there is no need any more to memorise or look up trigonometric sum and product equations. As an example, use complex numbers to show that:
$\Re e^{i(a+b)} = \cos(a+b) = \cos a \cos b - \sin a \sin b$

[3]When $y_\mathrm{r} \leq 0$ use the atan2 function with $\varphi = \mathrm{atan2}(y_\mathrm{i}, y_\mathrm{r})$, see `https://en.wikipedia.org/wiki/atan2`, or make a drawing and use trigonometry to determine φ.

[4]This holds for all analytic functions, but not for all non-analytic functions.

Exemplary solution

$$
\begin{aligned}
\Re e^{i(a+b)} &= \cos(a+b) = \Re[e^{ia} \times e^{ib}] \\
\Re[e^{ia} \times e^{ib}] &= \Re[(\cos a + i \sin a)(\cos b + i \sin b)] \\
&= \cos a \cos b - \sin a \sin b
\end{aligned} \tag{2.39}
$$

So, if you can solve this problem, you don't have to memorise any trigonometric sum, difference or double angle formula anymore, you can easily derive cosine and sine functions with complex numbers, taking the real or imaginary parts of $e^{i(a\pm b)}$ or $e^{2a} = e^{a+a}$.

Another useful property of complex numbers is that their real and imaginary parts behave as the x and y components of a vector in the xy-plane, where φ is the angle they make with the x axis. Thus complex numbers can be practical for adding planar vectors.

A **Example 2.8** Two force vectors $\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$ that both lie in the xy-plane act on a point A. The angles that the vectors make with the x axis are φ_1 and φ_2.

Problem: Calculate the absolute value of the total force vector $\vec{\boldsymbol{F}}_{\text{tot}} = \vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2$ acting on A.

Exemplary solution

We replace the vectors by the complex numbers F_{tot}, F_{c1} and F_{c2}, where we use that $\Re F_{c1} = F_{1,x}$ and $\Im F_{c1} = F_{1,y}$. Then we have:

$$
\begin{aligned}
|\vec{\boldsymbol{F}}_{\text{tot}}|^2 &= |F_{c1} + F_{c2}|^2 = (F_{c1} + F_{c2})(F_{c1} + F_{c2})^* \\
&= F_{c1}F_{c1}^* + F_{c2}F_{c2}^* + F_{c1}F_{c2}^* + F_{c2}F_{c1}^* \\
&= |F_{c1}|^2 + |F_{c2}|^2 + |F_{c1}|e^{i\varphi_1}|F_{c2}|e^{-i\varphi_2} + |F_{c2}|e^{i\varphi_2}|F_{c1}|e^{-i\varphi_1} \\
&= |F_{c1}|^2 + |F_{c2}|^2 + |F_{c1}||F_{c2}|[e^{i(\varphi_2-\varphi_1)} + e^{-i(\varphi_2-\varphi_1)}] \\
&= |\vec{\boldsymbol{F}}_1|^2 + |\vec{\boldsymbol{F}}_2|^2 + 2|\vec{\boldsymbol{F}}_1||\vec{\boldsymbol{F}}_2|\cos(\varphi_2 - \varphi_1)
\end{aligned} \tag{2.40}
$$

Taking the square root of this equation gives the absolute value of $|\vec{\boldsymbol{F}}_{\text{tot}}|$. Note that the equation we have derived is called the 'Law of cosines'. With complex numbers there is no need to memorise it.

A **Example 2.9** An important application of complex numbers is for analysing vibrations, as we will discuss in Ch. 13. A mass can vibrate at two angular frequencies ω_1 and ω_2 simultaneously, in which case its motion is written as: $x_{\text{tot}}(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$.

Problem: Show that this sum of trigonometric functions can also be written as a product of trigonometric functions: $x_{\text{tot}}(t) = 2\cos\left[\frac{1}{2}(\omega_1 + \omega_2)t\right] \cos\left[\frac{1}{2}(\omega_2 - \omega_1)t\right]$. This mathematical result leads to an important phenomenon called *beating* where the amplitude of vibrations, like sound, is modulated in time as shown in Fig. 2.1.

Exemplary solution

You can perform this derivation in two directions. We first start from the sum and prove that it is equal to the product:

$$\begin{aligned} x_{\text{tot}}(t) &= \Re[e^{i\omega_1 t} + e^{i\omega_2 t}] \\ &\quad \text{Now we multiply by } 1 = e^{i\frac{1}{2}(\omega_1+\omega_2)t} e^{-i\frac{1}{2}(\omega_1+\omega_2)t} \\ &= \Re\left[e^{i\frac{1}{2}(\omega_1+\omega_2)t}\left[e^{i\frac{1}{2}(\omega_1-\omega_2)t} + e^{i\frac{1}{2}(\omega_2-\omega_1)t}\right]\right] \\ &= \Re\left[e^{i\frac{1}{2}(\omega_1+\omega_2)t}\right] 2\cos\frac{1}{2}(\omega_2-\omega_1)t \\ &= 2\cos\left[\frac{1}{2}(\omega_1+\omega_2)t\right]\cos\left[\frac{1}{2}(\omega_2-\omega_1)t\right] \end{aligned} \tag{2.41}$$

Now we start from the given product and prove that it is equal to the sum:

$$\begin{aligned} x_{\text{tot}}(t) &= 2\cos\left[\frac{1}{2}(\omega_1+\omega_2)t\right]\cos\left[\frac{1}{2}(\omega_2-\omega_1)t\right] \\ &= \frac{1}{2}\left(e^{i\left[\frac{1}{2}(\omega_1+\omega_2)t\right]} + e^{-i\left[\frac{1}{2}(\omega_1+\omega_2)t\right]}\right)\left(e^{i\left[\frac{1}{2}(\omega_2-\omega_1)t\right]} + e^{-i\left[\frac{1}{2}(\omega_2-\omega_1)t\right]}\right) \\ &= \frac{1}{2}\left(\left[e^{i\left[\frac{1}{2}\times 2\omega_1 t\right]} + e^{-i\left[\frac{1}{2}2\omega_1 t\right]}\right] + \left[e^{i\left[\frac{1}{2}\times 2\omega_2 t\right]} + e^{-i\left[\frac{1}{2}2\omega_2 t\right]}\right]\right) \\ &= \cos(\omega_1 t) + \cos(\omega_2 t) \end{aligned} \tag{2.42}$$

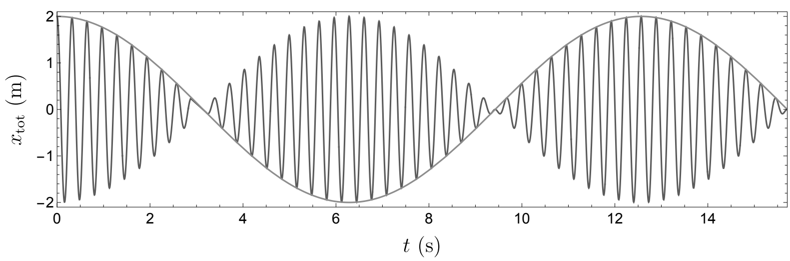


Figure 2.1: Plot of the function $x_{\text{tot}}(t) = \cos\omega_1 t + \cos\omega_2 t$ from example 2.9 and its envelope for ω_1=19 and ω_2=20. It is composed of the product of a high frequency cosine, with average frequency $\omega_{\text{avg}} = \frac{1}{2}(\omega_1 + \omega_2)$ and a low frequency cosine with beat frequency $\omega_{\text{beat}} = \frac{1}{2}(\omega_2 - \omega_1)$. The observed time varying amplitude of the high frequency cosine function is also called beating with $x_{tot}(t) = 2\cos(\omega_{\text{beat}} t)\cos(\omega_{\text{avg}} t)$.

2.6 How to solve problems

In general you can use the following flowchart [9] to solve problems in a structured way. The steps will be outlined in more detail in Ch. 14.

Follow these steps to solve a problem:

1. Make sure you *read the theory*, at least up to the point needed to solve the problem.
2. Read the entire problem description carefully, from beginning to end.
3. Write down a *list of variables* given and a second list of variables to be calculated.
4. Determine the type of problem.
5. Determine for yourself whether this is a problem of kinematics, kinetics, or other.
6. Break the problem down into sub-problems as appropriate and search for the required theory subsection in the book. Read this part again if needed.
7. Extract the general methods and formulae that need to be applied. Write down the equation number(s) from the book. Make sure you use the original, generic form of the formulae (not a similar-looking version from an example or a problem solution).
8. Make drawings, such as FBDs, or kinematic diagrams of your system. Use different views where needed. Clearly define at least one coordinate system including unit vector directions and a coordinate origin. Specify it if the coordinate system is moving or rotating.
9. Optionally create a table of variables where you clearly associate the variables in the problem and in your drawings with the variables given in the theoretic formulae.
10. Check all assumptions that need to hold for the formulae to be applicable
11. Write down all assumptions you made for your specific problem, and highlight those that can only be checked later.
12. In many problems you have to solve for one or more scalar variables. In general, you have E equations, and need to solve a system with U unknowns. To check if you have sufficient information (E) to solve the problem apply the following steps:

    a) Count the number U of scalar unknowns. Be careful to include all unique components of a vector or a matrix individually.
    b) Count the number E of scalar equations given by the equations you found.
    c) Compare E and U.

        - If $E = U$, continue.
        - If $E < U$, do not start solving yet. Instead, find more equations, either in the theory or in the problem itself. Are there kinematic relationships between variables that can reduce U or increase E?

- If $E > U$, continue, but make a note to come back to check if some of the equations do not contradict each other and if all of them are correct.

d) Solve the system of equations. Do this symbolically as much as possible, do not insert any numbers.
e) Check the original assumptions. Do they indeed hold? If not, go back to step 11, make other assumptions and solve the problem again.
f) Once you have found a result, conduct multiple plausibility checks. Also re-insert your solution into your original equations, especially if $E > U$. If in doubt about your result, start again at step 1.
g) If requested compute the numerical values of the U unknowns using the E obtained equations and the given numerical quantities.

2.7 Plausibility checks

A plausibility check is a calculation or validation procedure that is performed to verify if the obtained solution to a certain problem is correct.

The main takeaways from this section are the following:

- If your answer passes all plausibility checks, then the answer is probably correct.
- If your answer does *not* pass a plausibility check, then the answer is probably wrong and the failed check will likely provide you information where the mistake is, such that you can fix it.
- Therefore, if you learn properly to perform plausibility checks, you will not need a solution anymore to check if your answer is correct.
- Note that you can also perform a plausibility check at every intermediate step of the solution. This helps to find the step with the mistake.
- For professional engineers there is no solution manual. To avoid making mistakes, which can have a huge impact, they always need to check the plausibility of their calculations, to be close to 100% sure of their answer.
- Therefore checking the correctness of answers (without solution manual) is one of the most important skills an engineer should acquire.
- For this reason we do not give students answers to all problems, since if one uses a solution manual to verify the correctness of answers, one never learns how to properly perform plausibility checks.
- If you, despite the plausibility checks, do not manage to solve a problem, always ask explanation from your professor, teacher or colleagues.

Here are some examples of questions you can ask yourself to check the plausibility of an answer:

- Does the quantity in the answer have the right units?
- Are units on both sides of the equation equal? (Sec. 2.2)
- Is the sign of the answer correct when comparing it to the question? E.g. in a problem where a ball is falling downward due to gravity, and the y axis points upward, is v_y negative?
- Does the value of the answer make sense? (not more than 10 times smaller or larger than expected?). E.g. in a problem where a mass is accelerating due to gravity over a surface, its acceleration can never exceed the free fall value g.
- When you substitute the answer in the equations used, do you get back the given values? E.g. if you have found the solution $x(t) = c_1 t^2$, from the initially given acceleration, check by twice differentiating that $a(t) = 2c_1$.
- When projecting forces on axes, does the length of the vector correspond to that of the components? E.g. $v_x^2 + v_y^2 + v_z^2 = |\vec{\boldsymbol{v}}|^2$.
- Does the net force vector $\sum \vec{\boldsymbol{F}}$ point in the same direction as the acceleration $\vec{\boldsymbol{a}}$?
- Do the chosen axes form a right-handed coordinate system?
- Does $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and the other laws of Newton and Euler hold if you substitute the final answer back into them?
- Can methods of work and energy be used to double check the solution?
- Are all given constraint equations satisfied if you substitute the final answer into them?
- Is there any other information given that can be verified with the answer?

Of course there are many more plausibility checks that can be performed, which really depend on the type of problem under consideration.

3 Vectors and Coordinates

Since vectors and coordinate systems are essential for the description of dynamics, we discuss the definitions, conventions and basic operations (linear algebra and cross products) needed to deal with them in this chapter.

3.1 Vectors in mechanics

A vector (or Euclidean vector) is a unitless geometric object with a magnitude and direction in three-dimensional space. In science and engineering a vector can also have a unit. In this section we discuss several particular conventions and methods that are important for using and drawing vectors in mechanics.

3.1.1 Position vector

Probably the most important vector in dynamics is the position vector $\vec{\boldsymbol{r}}_A$ which represents the position of a point or point mass A in space by a straight arrow that points from the origin of the coordinate system to point A. Usually, a coordinate system is chosen such that the position vector of the origin has zero magnitude, $|\vec{\boldsymbol{r}}_O| = 0$.

3.1.2 Relative position vector

A relative position vector $\vec{\boldsymbol{r}}_{B/A}$ (see Fig. 3.1) is a vector that indicates the relative position of a point (or point mass) B with respect the position of another point A and is defined as:

$$\vec{\boldsymbol{r}}_{B/A} \equiv \vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A \tag{3.1}$$

Note that the order of the objects in the subscript is important: $\vec{\boldsymbol{r}}_{B/A}$ indicates the position of object B with respect to the position of A, and vice versa $\vec{\boldsymbol{r}}_{A/B}$ indicates the position of object A with respect to the position of B, inverting the direction of the vector, therefore $\vec{\boldsymbol{r}}_{B/A} = -\vec{\boldsymbol{r}}_{A/B}$, as is also seen from Eq. (3.1). A way to remember this is to replace the symbol / by the words 'with respect to'. So, $\vec{\boldsymbol{r}}_{B/A}$ is 'the position vector of point B with respect to point A'. A normal position vector (without / in the subscript) is always measured with respect to the origin O of the coordinate system.

The relative position vector is convenient to determine the shortest distance between two points in space. Relative velocity and acceleration vectors between two points are defined analogously:

$$\vec{\boldsymbol{v}}_{B/A} \equiv \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A \tag{3.2}$$

$$\vec{\boldsymbol{a}}_{B/A} \equiv \vec{\boldsymbol{a}}_B - \vec{\boldsymbol{a}}_A \tag{3.3}$$

Concept. *Distance*
The distance $d_{B/A}$ between two points A and B in space is given by the absolute value of the relative position vector connecting these two points.

$$d_{B/A} \equiv |\vec{\boldsymbol{r}}_{B/A}| = \sqrt{\vec{\boldsymbol{r}}_{B/A} \cdot \vec{\boldsymbol{r}}_{B/A}} \tag{3.4}$$

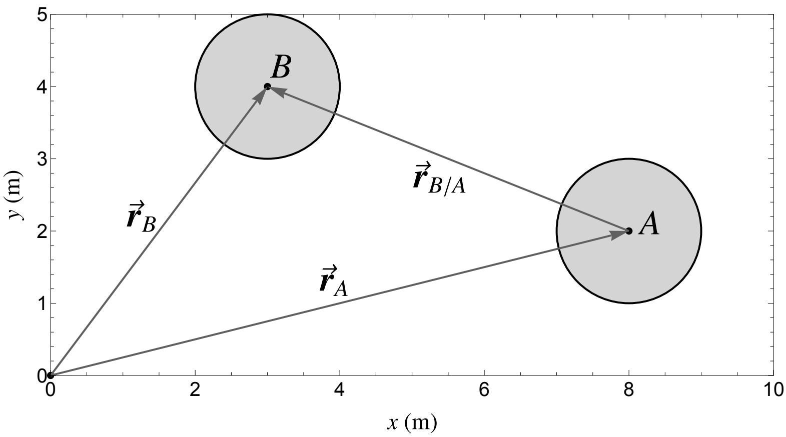


Figure 3.1: Absolute position vectors $\vec{\boldsymbol{r}}_A$, $\vec{\boldsymbol{r}}_B$, indicating the position of objects A and B with respect to the origin. Relative position vector $\vec{\boldsymbol{r}}_{B/A}$, indicating the position of object B with respect to the position of object A.

S **Example 3.1** Determine the distance between points A and B using the relative position vectors.

Exemplary solution
From Fig. 3.1 we find that: $\vec{\boldsymbol{r}}_A = (8\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})$m and $\vec{\boldsymbol{r}}_B = (3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}})$m. With Eq. (3.1) we find that $\vec{\boldsymbol{r}}_{B/A} = ([3-8]\hat{\boldsymbol{\imath}} + [4-2]\hat{\boldsymbol{\jmath}})$m. Then we obtain from Eq. (3.4) that the distance between points A and B is: $d_{B/A} = \sqrt{5^2 + 2^2}\,\text{m} = \sqrt{29}\,\text{m}$.

3.1.3 Vectors and unit vectors

Besides position vectors, we will deal with many other types of vectors for analysing dynamical systems, like velocity, angular velocity, force and torque vectors. In all these cases the vector, for instance a force $\vec{\boldsymbol{F}}$, expresses a magnitude and a direction. The magnitude of the vector can always be obtained by taking the absolute value (norm) of the vector:

$$|\vec{\boldsymbol{F}}| = \sqrt{\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{F}}} \tag{3.5}$$

The magnitude $|\vec{\boldsymbol{F}}|$ is a scalar with the unit newton (N). The direction of the force can be specified by a unit vector $\hat{\boldsymbol{F}}$, which is a vector with magnitude 1 that points in the same direction as $\vec{\boldsymbol{F}}$. That unit vector is defined as:

$$\hat{\boldsymbol{F}} \equiv \frac{\vec{\boldsymbol{F}}}{|\vec{\boldsymbol{F}}|} \tag{3.6}$$

Because they are a ratio of two quantities with the same unit, unit vectors, despite their name, have no unit. They are called 'unit' vector because they have a magnitude of unity). With these definitions any vector can be written as the product of a magnitude and a unit vector like in this example for the force vector:

$$\vec{\boldsymbol{F}} = |\vec{\boldsymbol{F}}|\hat{\boldsymbol{F}} \tag{3.7}$$

3.2 Drawing vectors

Although the motion of objects, can be fully expressed with mathematical equations, it is often much easier to visualise objects and mechanisms using drawings. Moreover, it is common practice in engineering to not provide all information in equation form, but provide part of it in the form of a drawing. Therefore, correctly drawing vectors, and interpreting those drawings is important for a correct analysis of dynamics. Analysing a problem in mechanics usually starts by making a sketch of the system, and later one or more free-body diagrams (FBDs), which are then translated into mathematical equations. To make sure no errors or inconsistencies occur during this translation process from drawing to mathematical equations, it is important to draw all vectors correctly and to have clear definitions on the way vectors are drawn and converted into equations.

A vector is always drawn as a straight arrow, with a label. Optionally, a dot can be placed at the tail or head of the arrow to indicate the point of action of the vector. If the arrow points perpendicular to the plane of the drawing, its direction can be indicated by the symbols ⊙ (pointing towards the reader out of the page) and ⊗ (pointing away from the reader into the page). The meaning of these symbols can be easily memorised by imagining a dart arrow that points in the same direction as the vector and realising that the symbols resemble the tip (⊙), or flight (⊗) of the dart arrow.

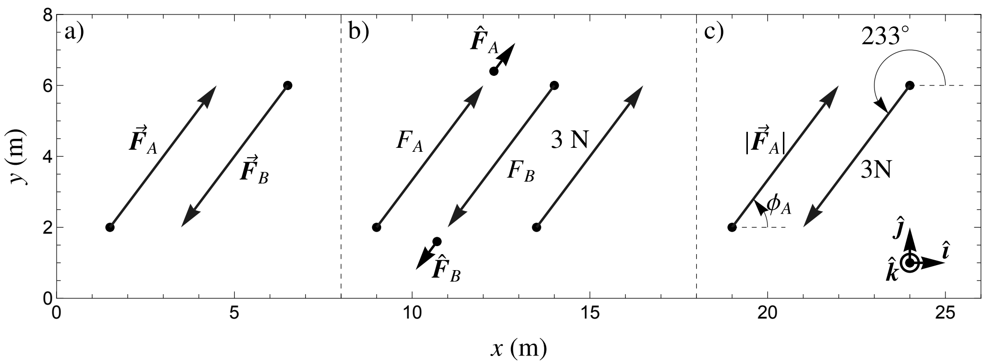


Figure 3.2: The three main ways of specifying vectors in a drawing. a) General vectors. b) Vectors specified by a scalar signed magnitude. c) Vectors specified by scalar magnitude and angle.

Figure 3.2 shows three main ways to specify vectors in a drawing:

1. Drawing general vectors $\vec{\boldsymbol{F}}_A$, $\vec{\boldsymbol{F}}_B$ (Fig. 3.2a).
2. Drawing vectors as an arrow with fixed direction and scalar signed magnitude F_A, F_B (Fig. 3.2b).
3. Drawing vectors with scalar magnitude $|F_A|$, $|F_B|$ and a direction specified by an angle ϕ_A, ϕ_B (Fig. 3.2c).

Although the first method of drawing without scalars can always be used, it is important to get used to the second and third method of drawing vectors as well, since they provide a way to use the drawing to provide part of the relevant information with less equations needed, and are therefore regularly used in engineering.

We note that in general, multiple 2D drawings, e.g. obtained by orthographic projection, are needed to visualise a 3D system. However, since in this textbook we will mainly deal with the dynamics in a single plane, one drawing is usually sufficient.

3.2.1 Drawing general vectors

General vectors are drawn by an arrow and a vector label like $\vec{\boldsymbol{F}}_A$ (Fig. 3.2a). When using this type of vector notation, the most important thing is to draw the vector with the right point of action in the drawing, to label it correctly and to make sure each relevant vector is only drawn once. Although it is convenient to draw the arrow in the direction in which the vector is expected to point, drawing it in the wrong direction will not cause problems, since both the direction and magnitude of the vector, and its conversion to scalars, are dealt with by the mathematics and a suitably chosen coordinate system. This means that in addition to the drawing an equation like $\vec{\boldsymbol{F}}_A = (3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}})$N is needed to fully specify the vector.

3.2.2 Drawing vectors with scalar signed magnitude

In some cases the direction of the line along which a vector operates is fixed, e.g. because a force operates along the line connecting two points in the drawing, but the magnitude of the vector is not, for instance because it is a time-dependent force vector or because it still needs to be calculated. In that case the vector can be represented by drawing an arrow in the direction the vector is known to act and adding a scalar label F_A next to it (Fig. 3.2b). In this case the correct drawing of the direction of the arrow is essential for the further analysis, because it defines a unit vector $\hat{\boldsymbol{F}}_A$ pointing in the same direction as the drawn arrow. Normally this unit vector is not drawn, but in this figure it is drawn to explain the concept of defining vectors with signed magnitudes. When using this method of drawing vectors, the scalar quantity F_A is defined as the *signed magnitude* of the vector $\vec{\boldsymbol{F}}_A$ and the drawing defines this vector by:

$$\vec{\boldsymbol{F}}_A \equiv F_A \hat{\boldsymbol{F}}_A \tag{3.8}$$

Note that the scalar value F_A can have a negative sign: that is the reason F_A is called a 'signed magnitude'. When F_A is negative, the vector $\vec{\boldsymbol{F}}_A$ points in the opposite direction as the drawn arrow and unit vector $\hat{\boldsymbol{F}}_A$. This also means that F_A can have a different sign than the vector's magnitude which is always positive ($F_A \pm |\vec{\boldsymbol{F}}_A|$). If the magnitude of the vector is known and fixed, then one can replace the label F_A by the magnitude of the vector including the unit, for instance by writing 3 N besides the arrow like in Fig. 3.2b).

3.2.3 Drawing vectors by magnitude and angle

A planar[1] vector $\vec{\boldsymbol{F}}_A$, can also be indicated by a magnitude $|\vec{\boldsymbol{F}}_A|$ and an angle ϕ_A (Fig. 3.2c). When specifying the angle ϕ_A, one also needs to specify the reference direction at which $\phi_A = 0$, which is indicated by a dashed line and is usually taken to be the positive x-axis. One also draws a circular arrow that points from the reference direction towards the drawn vector to indicate the direction in which the angle ϕ_A increases. In this textbook we use the convention that in the xy-plane such an angle has a positive scalar value if it rotates in the same direction as given by the right-hand rule[2]. So in the coordinate system drawn in Fig. 3.2 the angle increases in the counter-clockwise direction. Then, in a Cartesian coordinate system in the xy-plane (see later in this chapter) one finds that:

$$\vec{\boldsymbol{F}}_A = |\vec{\boldsymbol{F}}_A| \cos\phi_A \hat{\boldsymbol{\imath}} + |\vec{\boldsymbol{F}}_A| \sin\phi_A \hat{\boldsymbol{\jmath}} \tag{3.9}$$

An angle between two vectors can be specified as $\phi_{B/A} = \phi_B - \phi_A$. If the magnitude and/or angle of the vector are known and fixed, then one can replace the labels $|\vec{\boldsymbol{F}}_A|$ and ϕ_A by the magnitude and angle of the vector including the unit, for instance by writing '3 N' besides the arrow and replacing the angle ϕ_A by 233°, as can be seen in Fig. 3.2c). Inverting the direction of such a vector is done by adding π rad (=180°) to the angle ($\phi_A \rightarrow \phi_A + \pi$). This is a difference with the signed magnitude vector, for which the inversion is performed by changing the sign of the signed magnitude ($F_A \rightarrow -F_A$).

S **Example 3.2** Give the mathematical expressions for the following quantities such that the corresponding vectors in Fig. 3.2 have identical magnitude and direction as the vector in Fig. 3.2b) specified by label 3 N:

- $\vec{\boldsymbol{F}}_A$ and $\vec{\boldsymbol{F}}_B$ in Fig. 3.2a)
- F_A and F_B in Fig. 3.2b)
- The combination $|\vec{\boldsymbol{F}}_A|$ and ϕ_A in Fig. 3.2c)

Exemplary solution

- $\vec{\boldsymbol{F}}_A = \frac{3}{5}(3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}})$N and $\vec{\boldsymbol{F}}_B = \frac{3}{5}(3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}})$N
- $F_A = 3$ N and $F_B = -3$ N
- $|\vec{\boldsymbol{F}}_A| = 3$ N and $\phi_A = \arctan\frac{4}{3} \approx 53°$

[1] We only discuss in-plane vectors here, which can be specified with a single scalar angle. In general at least two angles are needed to specify a vector in 3D.

[2] So by putting the thumb of one's right-hand along the z-axis, the direction of positive increasing angle is that in which the curved fingers point.

3.2.4 Drawing fixed, sliding and free vectors

A vector only has a direction and magnitude. However, for vectors in mechanics, the position at which vectors are drawn can be essential for the correct analysis. For instance the position of the origin is essential for the interpretation of a position vector, and the point of action of a force vector can be essential for its effect on a rigid body. For this reason, vectors can be complemented by a point (reference point, or point of action), or by a line of points (line of action). The combination of the vector and this point or line is then called a fixed vector or a sliding vector.

1. **Fixed vectors:** these are vectors which have to be drawn at a fixed *point of action* or reference point for the correct dynamic analysis. This point of action or reference point should be indicated by a black dot on the tail or tip of the arrow. Position vectors are fixed vectors and always have their point of reference at the tail of the arrow. The vector $\vec{\boldsymbol{r}}_A$ in Fig. 3.3 is an example.
2. **Sliding vectors:** for sliding vectors, instead of a point of action, one can use a *line of action*, that is shown as a dashed line in Fig. 3.3. The line of action of a force vector $\vec{\boldsymbol{F}}_B$ is the line that passes through its point of action B and is parallel to $\vec{\boldsymbol{F}}_B$. For sliding vectors any point of action that lies on the line of action can be chosen without affecting the dynamic analysis, so the vector can slide along the line. For example, to determine the moment vector of force $\vec{\boldsymbol{F}}_B$ with respect to a reference point P, one uses the cross product $\vec{\boldsymbol{M}}_{B/P} = \vec{\boldsymbol{r}}_{B/P} \times \vec{\boldsymbol{F}}_B$. It can be shown that replacing point B by any other point on the *line of action*, results in the same moment vector $\vec{\boldsymbol{M}}_{B/P}$. This is visualised in Fig. 3.3 by showing that the area of the red rectangle and red parallelogram are equal. When sliding a vector, the line of action should be drawn as a dashed line that passes through the point of action or reference point, which should still be indicated by a black dot, like point B in Fig. 3.3. Sliding a vector along its line of action can be performed to facilitate or clarify a drawing, but is in particular useful to facilitate determining a cross product. As will be discussed in Sec. 3.4, the magnitude of such a cross product $\vec{\boldsymbol{r}}_{B/P} \times \vec{\boldsymbol{F}}_B$ is equal to the area of the red parallelogram Fig. 3.3. As shown by the red rectangle in Fig. 3.3, one can slide the tail of the vector $\vec{\boldsymbol{F}}_B$ along the line of action to the point nearest to P at a distance $r_{\min}$. Then the cross product is simply proportional to the area of the red rectangle, so $|\vec{\boldsymbol{M}}_{B/P}| = r_{\min}|\vec{\boldsymbol{F}}_B|$.
3. **Free vectors:** There is no reference point for a free vector, so free vectors can be drawn anywhere. However it is often clearer to draw them

on or near the object on which they act, like shown for the vector $\vec{\boldsymbol{M}}_{B/P}$ in Fig. 3.3. The moment, couple, and angular momentum vector are free vectors. Also the angular velocity and angular acceleration vectors of a rigid body are examples of free vectors.

4. **Vector fields:** vector fields consist of vectors at every position in space. They are best drawn as vectors on the objects on which they act, while their spatial dependence is given as equations. Alternatively vector fields can be drawn and visualised at many points in space (similar to Fig. 5.6). Since the gravitational acceleration vector field on earth is approximately constant, it can be drawn at a single position like shown in Fig. 3.3 using signed magnitude notation $\vec{\boldsymbol{g}} = -g\hat{\boldsymbol{\jmath}}$, by which we indicate that it is identical for all objects in the figure.

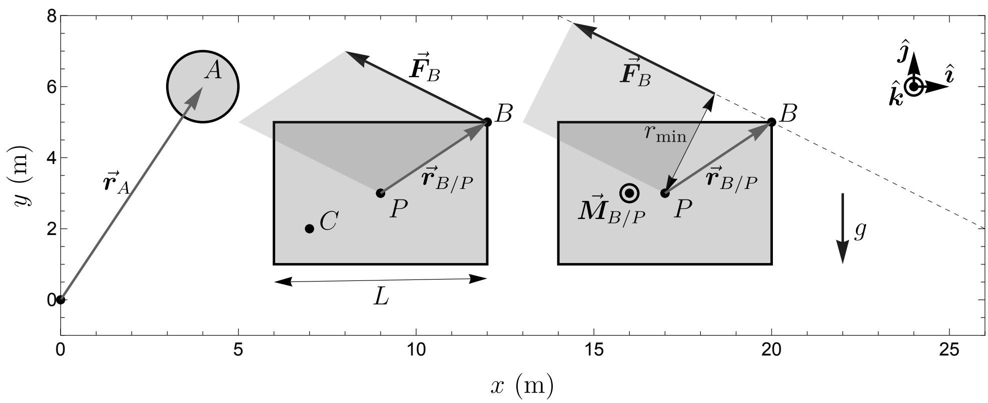


Figure 3.3: Examples of drawing fixed, sliding and free vectors, and distances. The areas of the red parallelogram and red rectangle are equal to the magnitude of the cross product $|\vec{\boldsymbol{r}}_{B/P} \times \vec{\boldsymbol{F}}_B|$.

3.2.5 Projecting vectors and drawing components

After drawing the vectors and defining the coordinate system, the vectors can be projected on the coordinate axes to obtain and draw the vector components. To obtain the components along each of the axes, we take the dot product with the unit vectors. For example in Cartesian coordinates we get for the projection of the vector $\vec{\boldsymbol{F}}_A$ on the coordinate system (CS) the following 3

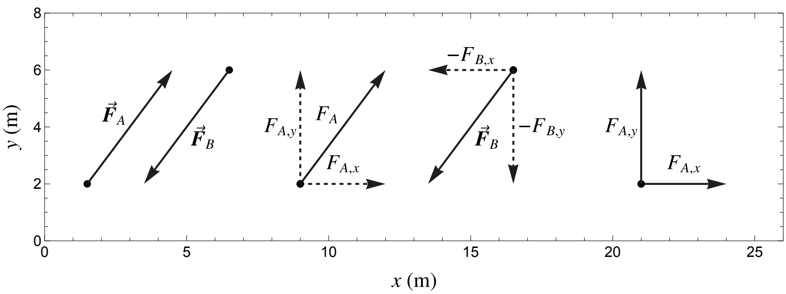


Figure 3.4: Drawing components of vectors $\vec{F}_A$ and $\vec{F}_B$, projected on Cartesian coordinate axes. If you draw both the vector and its components, use dashed lines for the components.

components:

$$F_{A,x} = \vec{F}_A \cdot \hat{\imath} \tag{3.10}$$

$$F_{A,y} = \vec{F}_A \cdot \hat{\jmath} \tag{3.11}$$

$$F_{A,z} = \vec{F}_A \cdot \hat{k} \tag{3.12}$$

As a result we have $\vec{F}_A = F_{A,x}\hat{\imath} + F_{A,y}\hat{\jmath} + F_{A,z}\hat{k}$, and projected vector components can be considered as vectors themselves, e.g. $\vec{F}_{A,x} = F_{A,x}\hat{\imath}$, with axis subscripts x, y, z to indicate the projected components. Examples of drawing projected vector components are shown in Fig. 3.4. Projected vector components are in practice always drawn using signed magnitude notation (so with scalar labels). If both the vector and its projected vector components are drawn, one has to use dashed arrows for the drawn vector components, to distinguish them from the original vector (see examples in Fig. 3.4). This is needed to prevent errors from double counting of forces in an FBD. According to the rules for drawing vectors with signed magnitude notation, a minus sign is needed in front of the projected components when the arrow points in the negative axis direction, like for $-F_{B,x}$ in Fig. 3.4.

3.2.6 Scaling of drawn vectors

If you want to use geometric techniques to make calculations with vectors, then it is important to draw them with the correct magnitude. Euclidean vectors are drawn in a coordinate system with axes with distance dimensions and units in e.g. meters. Using these axes, the magnitude of a position vector can be determined by measuring its length. However, the magnitude of other vectors, like velocity or force (see e.g. vector $\vec{F}_B$ in Fig. 3.3), that have units of m/s or N cannot be determined from the axes. There is therefore always a scale factor that can be chosen arbitrarily to convert the drawn length of such a vector to its actual magnitude. For example a force vector $\vec{\boldsymbol{F}}$ that has a magnitude $|\vec{\boldsymbol{F}}| = 1\,\mathrm{N}$ can be drawn with a length of 1 m. In that case there is a scale factor for force vectors $\gamma_F = 1\,\mathrm{m/N}$, but other scale factors might also be chosen. After choosing a scale factor, the drawn length of all force vectors should be determined from their magnitude using the equation $\gamma_F|\vec{\boldsymbol{F}}|$. This scaling is especially important when graphical methods are used that use the drawn magnitudes of vectors.

3.2.7 Drawing points

Points are drawn as black dots with a label in a sketch or FBD. It is important to realise that there are different types of points. There are points that are fixed[3] in space, but also points that move along with a rigid body, like the point B in Fig. 3.3. To clarify in a drawing which points move along with a rigid body and which ones do not, we adopt the following conventions in this textbook:

- Relevant points are always indicated by a black dot.
- If the black dot is not touching a rigid body, the point is assumed to be fixed space, like the point at $(0, 0)$ in Fig. 3.3.
- If the black dot is touching a rigid body or its edge, the point is moving along with the rigid body (like any mass inside the rigid body), like point B and C in Fig. 3.3.
- In all other cases more information needs to be provided to clarify if the point indicated by the drawn dot is fixed in space or moving along with one of the objects in the drawing. As an example see the caption of Fig. 9.8, where the drawn dots touch 2 rigid bodies, and different labels are used to indicate points that move along with different rigid bodies, while being represented by the same black dot.

The main message is to be aware that there are different types of points in dynamics: points fixed in space, and points fixed to rigid bodies or other objects. Further clarification of the motion of points can be provided in a text that accompanies the drawing.

[3]To be more precise, with the words ’fixed in space’, we mean that it has zero velocity and acceleration in the current reference frame. With ’move along with a rigid body’ we mean that the point has zero velocity and acceleration in a reference frame for which all points of the rigid body have zero velocity and acceleration. We will discuss reference frames in more detail in Sec. 6.12.

3.2.8 Drawing distances and dimensions

Often dimensions and distances (see Sec. 3.1.2) between two points needs to be drawn. Distances are always positive and drawn as a double sided arrow, with arrow heads on both of its ends and/or small perpendicular end caps. An example is shown in Fig. 3.3 for the distance L that characterises the side length of the rectangle.

3.3 Coordinate systems

In this textbook we deal with two main types of coordinate systems for determining position and analyse motion in three dimensional (3D) space: Cartesian coordinates and cylindrical coordinates. We will not discuss spherical coordinates.

3.3.1 Cartesian coordinates

In Cartesian coordinates, space is spanned by x, y and z coordinate axes, which determine the position of objects with respect to the origin O, as is shown in Fig. 3.5. Along each of the axes a unit vector with length 1 is defined, respectively $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$, which point in the positive axis directions.

Note. *When drawing a coordinate system in 3 dimensions, the convention is to always use right-handed axis system, since otherwise the vector rules for taking cross-products will fail. For Cartesian coordinates this means that you should make sure, e.g. using the right-hand rule, that for a fixed choice of the order of coordinates* x, y, z, *we have the relation* $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}}$ *between the respective unit vectors.*

To describe the position vector $\vec{\boldsymbol{r}}_A$ of a point mass A with respect to the origin in Cartesian coordinates you can use one of the following 3 notations:

$$\vec{\boldsymbol{r}}_A = x_A\hat{\boldsymbol{\imath}} + y_A\hat{\boldsymbol{\jmath}} + z_A\hat{\boldsymbol{k}} \tag{3.13}$$

$$= \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = \begin{bmatrix} x_A & y_A & z_A \end{bmatrix}^{\mathrm{T}} \tag{3.14}$$

Where the superscript T in (3.14) stands for the transpose of the matrix, converting the row vector to a column vector. We use square brackets and no commas to indicate vectors, and round brackets with commas for the coordinates (x_A, y_A, z_A).

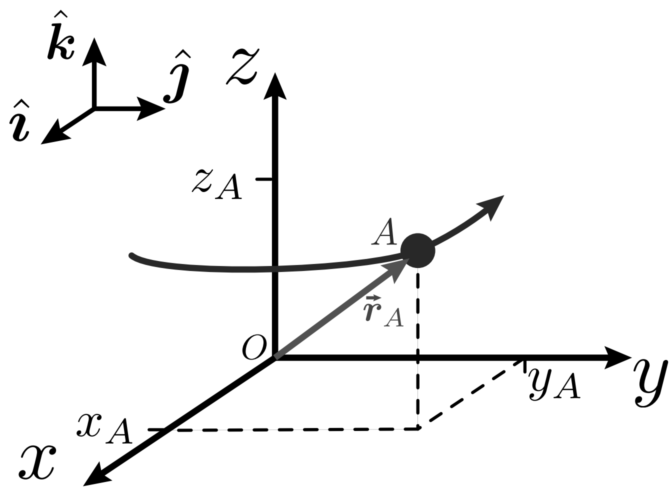


Figure 3.5: Point mass A, at position vector $\vec{\boldsymbol{r}}_A = x_A\hat{\boldsymbol{\imath}} + y_A\hat{\boldsymbol{\jmath}} + z_A\hat{\boldsymbol{k}}$, moving in a Cartesian coordinate system.

3.3.2 Cylindrical coordinates

Cylindrical coordinates (ρ, ϕ, z) are often useful, for example to analyse the dynamics of a point mass A that rotates around an axis (the z-axis). An example of such a coordinate system in 2D is shown in Fig. 3.6 and in 3D Fig. 3.7. Ideally the z-axis of the coordinate system is chosen parallel to the rotation axis, and the origin of the coordinate system is chosen to be located at a suitable position on the axis. Then the position of the point mass A can be described by 3 coordinates: the radial coordinate ρ_A that determines the shortest distance to the axis, the azimuthal angle ϕ_A with respect to a reference line (polar axis) and the axial coordinate z_A. The unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{k}}$, which are shown in the figure, point in the direction the tip of the position vector would move if you would increase 1 of the coordinate values by a small amount. Mathematically the unit vector at the position $\hat{\boldsymbol{r}}_A$ is $\hat{\boldsymbol{\phi}}_A = \frac{\partial \vec{\boldsymbol{r}}_A}{\partial \phi_A} / |\frac{\partial \vec{\boldsymbol{r}}_A}{\partial \phi_A}|$, and similarly for $\hat{\boldsymbol{\rho}}_A$ and $\hat{\boldsymbol{k}}_A$. In contrast to Cartesian vectors, the direction of unit vectors in cylindrical coordinates can depend on the position vector $\vec{\boldsymbol{r}}_A$ of the object. So if there are multiple objects it can be useful to label the relevant unit vector with a subscript A.

Note. *Like in Cartesian coordinates one should adhere to the convention that a cylindrical coordinate system ρ, ϕ, z is a right-handed axis system with $\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{k}}$, by choosing ϕ to increase in the anticlockwise direction, as observed when looking from the positive z-axis towards the origin. This is consistent*

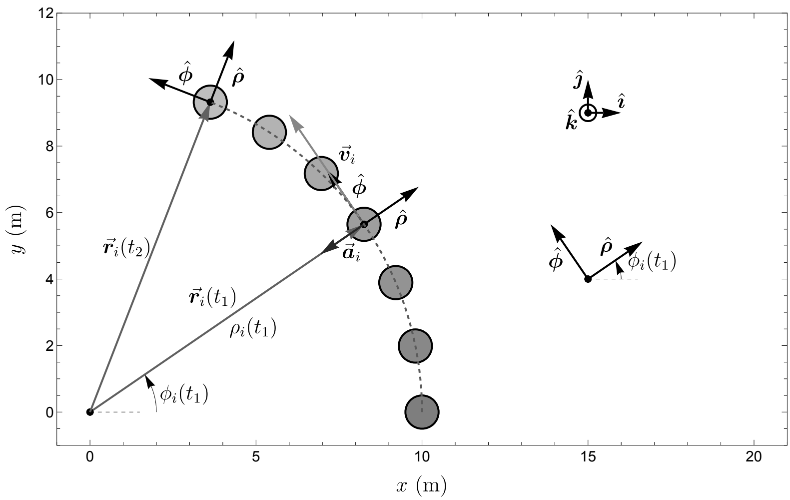


Figure 3.6: Point mass i moving in the xy-plane. Its position can be indicated by cylindrical coordinates ρ_i, ϕ_i and z_i. The directions of the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ depend on the position of the mass.

with the convention to take the anticlockwise direction as the positive direction for measuring angles.

A force vector $\vec{\boldsymbol{F}}_A$ that acts on a particle A at cylindrical coordinates (ρ_A, ϕ_A, z_A) can be expressed in the following 3 ways:

$$\vec{\boldsymbol{F}}_A = F_{A\rho}\hat{\boldsymbol{\rho}} + F_{A\phi}\hat{\boldsymbol{\phi}} + F_{Az}\hat{\boldsymbol{k}} \tag{3.15}$$

$$= \begin{bmatrix} F_{A\rho} \\ F_{A\phi} \\ F_{Az} \end{bmatrix} = \begin{bmatrix} F_{A\rho} \ F_{A\phi} \ F_{Az} \end{bmatrix}^T \tag{3.16}$$

One needs to be careful when working with vectors in cylindrical coordinates, because the vector component values depend on the coordinates (ρ_A, ϕ_A, z_A). Standard vector operations, like addition, between multiple vectors in cylindrical coordinates can therefore only be carried out if the vectors refer to the same coordinates, such that their unit vectors are identical. In other cases the vectors should first be converted to Cartesian coordinates, which can be done using the following relation between the unit vectors:

$$\hat{\boldsymbol{\imath}} = \cos\phi_A \ \hat{\boldsymbol{\rho}} - \sin\phi_A \ \hat{\boldsymbol{\phi}} \tag{3.17}$$

$$\hat{\boldsymbol{\jmath}} = \sin\phi_A \ \hat{\boldsymbol{\rho}} + \cos\phi_A \ \hat{\boldsymbol{\phi}} \tag{3.18}$$

$$\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \tag{3.19}$$

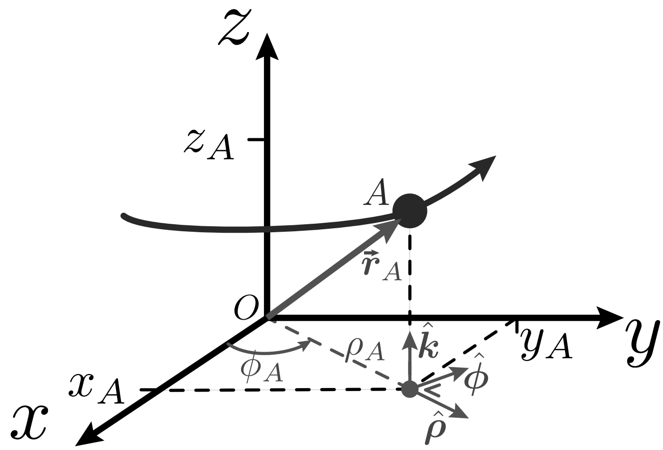


Figure 3.7: Point mass A moving in a cylindrical coordinate system. The directions of the ρ and ϕ coordinate axes (indicated in green), and the corresponding unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ depend on the position of the mass.

With these relations the components of a Cartesian vector like $\vec{\boldsymbol{F}}_A$ can be related to its cylindrical vector components $F_{A\rho}$, $F_{A\phi}$ and F_{Az} by taking its dot product with the unit vectors:

$$\vec{\boldsymbol{F}}_A = \begin{bmatrix} F_{Ax} \\ F_{Ay} \\ F_{Az} \end{bmatrix} = \begin{bmatrix} \vec{\boldsymbol{F}}_A \cdot \hat{\boldsymbol{\imath}} \\ \vec{\boldsymbol{F}}_A \cdot \hat{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{F}}_A \cdot \hat{\boldsymbol{k}} \end{bmatrix} = \begin{bmatrix} F_{A\rho} \cos\phi_A - F_{A\phi} \sin\phi_A \\ F_{A\rho} \sin\phi_A + F_{A\phi} \cos\phi_A \\ F_{Az} \end{bmatrix}, \tag{3.20}$$

where we replaced the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ by Eq. (3.17)–(3.19).

3.3.3 Position vector in cylindrical coordinates

It is important to note that whereas in Cartesian coordinates the components of the position vector $\vec{\boldsymbol{r}}_A = x_A\hat{\boldsymbol{\imath}} + y_A\hat{\boldsymbol{\jmath}} + z_A\hat{\boldsymbol{k}}$ are identical to the coordinates x_A, y_A, z_A, the position vector in cylindrical coordinates is given by $\vec{\boldsymbol{r}}_A = \rho_A\hat{\boldsymbol{\rho}} + z_A\hat{\boldsymbol{k}}$ and does not provide information about the angle ϕ_A. Motion in cylindrical coordinates can therefore better by expressed by giving the 3 coordinate functions $(\rho_A(t), \phi_A(t), z_A(t))$ than using the position vector.

$$\vec{\boldsymbol{r}}_A = \rho_A \hat{\boldsymbol{\rho}} + z_A \hat{\boldsymbol{k}} \tag{3.21}$$

$$= \begin{bmatrix} \rho_A \\ 0 \\ z_A \end{bmatrix} \neq \begin{bmatrix} \rho_A \\ \phi_A \\ z_A \end{bmatrix} \tag{3.22}$$

It is incorrect to have ϕ_A in the vector notation as shown on the right side of Eq. (3.22), as is also evident from the units which should be equal for every vector component.

3.4 Vector products

To mathematically express the laws of dynamics, we will often need two types of vector products: the *dot product* indicated by the symbol $\cdot$, and the *cross product*, indicated by the symbol $\times$. Note that these symbols only represent dot and cross products when they are placed between two vectors. If they are placed between two scalars or a scalar and a vector they indicate scalar multiplication.

3.4.1 Dot products using unit vector notation

Dot products or cross products of vectors can be evaluated using unit vector notation, using the following relations for dot products between unit vectors, that follow from their orthogonality:

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1 \tag{3.23}$$

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\imath}} = 0 \tag{3.24}$$

All dot products can be evaluated using these relations between the unit vectors. For example one now can take the dot product between two vectors $\vec{\boldsymbol{v}} = v_x \hat{\boldsymbol{\imath}} + v_y \hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{F}} = F_x \hat{\boldsymbol{\imath}} + F_y \hat{\boldsymbol{\jmath}}$ as follows:

$$\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{F}} = (v_x \hat{\boldsymbol{\imath}} + v_y \hat{\boldsymbol{\jmath}}) \cdot (F_x \hat{\boldsymbol{\imath}} + F_y \hat{\boldsymbol{\jmath}}) \tag{3.25}$$

$$= v_x F_x \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} + v_y F_x \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\imath}} + v_x F_y \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} + v_y F_y \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} \tag{3.26}$$

$$= v_x F_x \times 1 + v_y F_x \times 0 + v_x F_y \times 0 + v_y F_y \times 1 \tag{3.27}$$

$$= v_x F_x + v_y F_y \tag{3.28}$$

So you just multiply the scalar values by the dot products of the unit vectors. Here we used Eq. (3.23) and Eq. (3.24) to determine the dot products between the unit vectors like $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}$.

3.4.2 Cross product using unit vector notation

For cross products the relations between unit vectors in a right-handed axes system are as follows:

$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \vec{\mathbf{0}} \tag{3.29}$$

$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{k}} \tag{3.30}$$

$$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{\imath}} \tag{3.31}$$

$$\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = -\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\jmath}} \tag{3.32}$$

Where we used that for every cross product it holds that $\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}} = -\vec{\boldsymbol{B}} \times \vec{\boldsymbol{A}}$ (whereas for dot products $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} = \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{A}}$). With these relations all cross products can be evaluated using the unit vectors. For example one now can take the cross product between two vectors $\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$ as follows:

$$\begin{aligned} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}} &= (r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}) \times (F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}) \\ &= r_xF_x\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} + r_yF_x\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}} + r_xF_y\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} + r_yF_y\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} \\ &= r_xF_x \cdot \vec{\mathbf{0}} + r_yF_x \cdot (-\hat{\boldsymbol{k}}) + r_xF_y \cdot (+\hat{\boldsymbol{k}}) + r_yF_y \cdot \vec{\mathbf{0}} \\ &= (r_xF_y - r_yF_x)\hat{\boldsymbol{k}} \end{aligned} \tag{3.33}$$

Where we used Eq. (3.29) and Eq. (3.30) to evaluate the cross products between the unit vectors. Using unit vector notation has several advantages. You only have to write down and evaluate those unit vectors that are non-zero, the notation takes much less space and it is immediately clear what type of coordinate system one is working in from the unit vectors. For these reasons we will usually work with unit vector notation in this textbook.

3.4.3 Graphical analysis of dot and cross products

Dot and cross products can also be evaluated graphically as is shown in Fig. 3.8. The two vectors are drawn such that their tails are at the same point in space and the planar angle $\phi_{B/C}$ between them is determined. To obtain the dot product one projects one of the vectors on the direction of the other, obtaining the scalar $|F_B| \cos \phi_{B/C}$, which is then multiplied by the magnitude of the other vector $|v_C|$ giving the result:

$$\vec{\boldsymbol{v}}_C \cdot \vec{\boldsymbol{F}}_B = |\vec{\boldsymbol{v}}_C||\vec{\boldsymbol{F}}_B| \cos \phi_{B/C}. \tag{3.34}$$

The magnitude of this scalar value is proportional to the area of the pink parallelogram shown in Fig. 3.8. Note that the direction of the distance $|v_C|$ should be drawn at 90° to the actual vector $\vec{\boldsymbol{v}}_C$ to construct this parallelogram for the dot product.

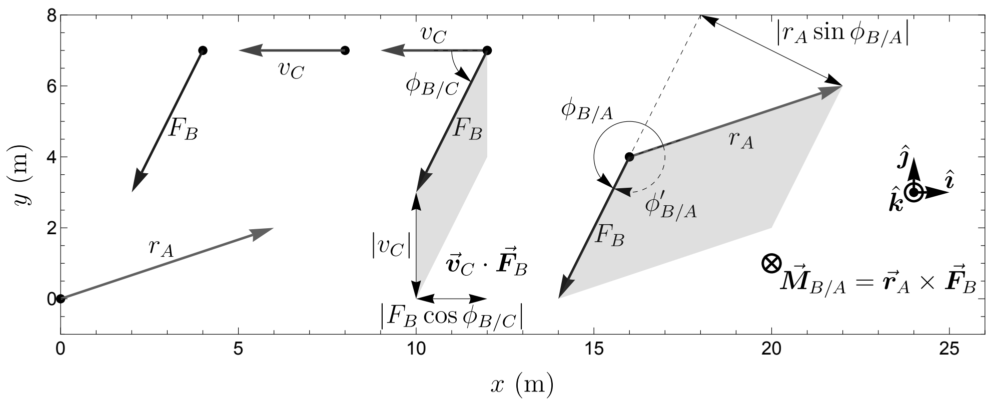


Figure 3.8: Demonstration of how to graphically determine the dot product $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{F}}_B$ and the cross product $\vec{\boldsymbol{M}}_{B/A} = \vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{F}}_B$. The areas of the light-red parallelograms indicate the magnitude of the dot and cross product, note that a line $|v_C|$ is drawn perpendicular to the vector $\vec{\boldsymbol{v}}_C$ to visualise its dot product with vector $\vec{\boldsymbol{F}}_B$.

For graphically analysing the cross product of two vectors $\vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{F}}_B$ (see Fig. 3.8), one puts their tails together and draws a circular angle arrow $\phi_{B/A}$, that starts at the first vector $\vec{\boldsymbol{r}}_A$ (left of the $\times$) and points anticlockwise towards the second vector $\vec{\boldsymbol{F}}_B$. Then if both vectors are in the xy-plane, their cross product can be determined as:

$$\vec{\boldsymbol{M}}_{B/A} = \vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{F}}_B = |\vec{\boldsymbol{r}}_A||\vec{\boldsymbol{F}}_B| \sin \phi_{B/A} \hat{\boldsymbol{k}} \tag{3.35}$$

As can be seen in Fig. 3.8, the magnitude of the cross product vector is equal to the red parallelogram whose sides are formed by the two vectors. The angle $\phi_{B/A}$ is measured in the anticlockwise direction. Since $\phi_{B/A} > 180°$, we have $\sin \phi_{B/A} < 0$ and the vector $\vec{\boldsymbol{M}}_{B/A}$ points in the $-\hat{\boldsymbol{k}}$ direction, into the plane as indicated by the $\otimes$ sign.

Right-hand rule

The direction of the cross-product vector $\vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{F}}_B$ can also conveniently be determined using the right-hand rule. In that case it is important not to always draw an anticlockwise angle, but to draw the *smallest* angle $\phi'_{B/A}$ from the first vector $\vec{\boldsymbol{r}}_A$ in the cross product to the second vector $\vec{\boldsymbol{F}}_B$ (see dashed arrow $\phi'_{B/A}$ in Fig. 3.8). So, make sure $\phi'_{B/A} < 180°$. Then curve the fingers of your *right hand* parallel to that curved arrow, with fingers pointing in the same direction as the arrowhead. Your thumb will then point in the same direction as $\vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{F}}_B$ in the $-\hat{\boldsymbol{k}}$ direction. It is recommended to always check the direction of vectors resulting from cross-product calculations using the right-hand rule. There are several alternative variations of right-hand and left-hand rules to determine the direction of a cross-product. Feel free to choose and memorise the one that you find most easy to use.

S **Example 3.3**

a) Determine the values of the dot product $\vec{\boldsymbol{v}}_C \cdot \vec{\boldsymbol{F}}_B$ and the cross product $\vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{F}}_B$ using unit vector notation. Use that the length of the vectors in Fig. 3.8 is scaled such that 1 m : 1 m/s : 1 N.
b) Determine the values of the dot product $\vec{\boldsymbol{v}}_C \cdot \vec{\boldsymbol{F}}_B$ and the cross product $\vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{F}}_B$ using graphical analysis.

Exemplary solution

a) Since all vectors are in the xy-plane we can reuse Eq. (3.28) and use Fig. 3.8 to obtain:

$$\vec{\boldsymbol{v}}_C \cdot \vec{\boldsymbol{F}}_B = v_{C,x}F_{B,x} + v_{C,y}F_{B,y} \quad (3.36)$$
$$= [(-3 \times -2) + (0 \times -4)] \text{ N} \cdot \text{m/s} \quad (3.37)$$
$$= 6 \text{ N} \cdot \text{m/s} \quad (3.38)$$

For the cross product between two vectors in the xy-plane we can use the result from Eq. (3.33):

$$\vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{F}}_B = (r_{A,x}F_{B,y} - r_{A,y}F_{B,x})\,\hat{\boldsymbol{k}} \quad (3.39)$$
$$= (6 \times (-4) - 2 \times (-2)) \text{ N} \cdot \text{m}\,\hat{\boldsymbol{k}} \quad (3.40)$$
$$= -20 \text{ N} \cdot \text{m}\,\hat{\boldsymbol{k}} \quad (3.41)$$

b) First we determine the angles of the vectors with respect to the positive x-axis direction. We have $\phi_A = \arctan\frac{2}{6} = 18.4°$, $\phi_B = 180° + \arctan\frac{-4}{-2} = 243.4°$ and $\phi_C = \arctan\frac{0}{-3} = 180°$. Then we find $\phi_{B/C} = \phi_B - \phi_C = 63.4°$ and $\phi_{B/A} = \phi_B - \phi_A = 225°$. We determine the magnitude of the three vectors: $|\vec{\boldsymbol{r}}_A| = \sqrt{r_{A,x}^2 + r_{A,y}^2} =$

$\sqrt{40}\,\mathrm{m}$, $|\vec{\boldsymbol{F}}_B| = \sqrt{F_{B,x}^2 + F_{B,y}^2} = \sqrt{20}\,\mathrm{N}$, $|\vec{\boldsymbol{v}}_C| = \sqrt{v_{C,x}^2 + v_{C,y}^2} = 3\,\mathrm{m/s}$. Filling these numbers in Eq. (3.34) and Eq. (3.35) we find:

$$\vec{\boldsymbol{v}}_C \cdot \vec{\boldsymbol{F}}_B = 3\,\mathrm{m/s} \times \sqrt{20}\,\mathrm{N}\ \cos(63.4^\circ) \tag{3.42}$$
$$= 6\ \mathrm{N \cdot m/s} \tag{3.43}$$
$$\vec{\boldsymbol{r}}_A \times \vec{\boldsymbol{F}}_B = \sqrt{40}\,\mathrm{m} \times \sqrt{20}\,\mathrm{N}\ \sin(225^\circ)\ \hat{\boldsymbol{k}} \tag{3.44}$$
$$= -20\ \mathrm{N \cdot m}\ \hat{\boldsymbol{k}} \tag{3.45}$$

So, as it should be, both the unit vector and graphical method to determine dot and cross-products give the same result.

3.5 Vector functions, fields and calculus

Vectors can be parametrized, such that their values depend on a scalar parameter like time t or position on a curve s. Such a parametrized vector, like $\vec{\boldsymbol{F}}(t)$ or $\vec{\boldsymbol{F}}(s)$ is also called a vector function. If a coordinate system is chosen, the vector function can be described by three scalar functions, $\vec{\boldsymbol{F}}(t) = F_x(t)\hat{\boldsymbol{\imath}} + F_y(t)\hat{\boldsymbol{\jmath}} + F_z(t)\hat{\boldsymbol{k}}$.

3.5.1 Vector and scalar fields

Even though they do not play an important role in this textbook we briefly mention the concept of *vector fields.* A vector field is a vector function that is defined at any point in (a region of) space. This is done by using the coordinates as parameters of the vector function, like in $\vec{\boldsymbol{F}}(x, y, z)$. Examples of vector fields are the gravitational force field $\vec{\boldsymbol{F}}(x, y, z)$ acting on a planet and the velocity field $\vec{\boldsymbol{v}}(x, y, z)$ describing the velocities of all point masses in a rigid body. Instead of providing a single vector at one position, a vector field describes a field of vectors that are present at every location in space. Vector fields can also be time-dependent like in $\vec{\boldsymbol{F}}(x, y, z, t)$. One can also define scalar fields, like $V(x, y, z)$ which define a scalar value at every position. We will discuss scalar fields in more detail when dealing with potential energy in Ch. 7.

3.5.2 Calculus on vector functions

Vector calculus deals with methods to integrate or differentiate vector functions and fields. Although a complete discussion of vector calculus is outside the scope of this textbook, we say a few words about it here. Differentiation and integration over a vector functions can be performed by separating the vector into its components and then proceeding with each component like with scalar functions, e.g. for a time integral $\int \vec{\boldsymbol{F}}(t)\mathrm{d}t = \int F_x(t)\hat{\boldsymbol{\imath}}\mathrm{d}t + \int F_y(t)\hat{\boldsymbol{\jmath}}\mathrm{d}t +$

$\int F_z(t)\hat{\boldsymbol{k}}\mathrm{d}t$. Taking derivatives is performed similarly $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{F}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}F_x(t)\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}}{\mathrm{d}t}F_y(t)\hat{\boldsymbol{\jmath}} + \frac{\mathrm{d}}{\mathrm{d}t}F_z(t)\hat{\boldsymbol{k}}$. Often Cartesian coordinates are to be preferred, because these operations become quite difficult in e.g. cylindrical coordinates where the unit vectors depend on position. However, when discussing kinematics we will deal with time derivatives of vector functions in cylindrical coordinates.

3.5.3 Calculus on scalar and vector fields

The differential operator, signified by the nabla symbol $\vec{\boldsymbol{\nabla}}$, is important for analysing scalar and vector fields. In Cartesian coordinates it is defined as $\vec{\boldsymbol{\nabla}} = \frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}$. The main operations that can be performed using $\vec{\boldsymbol{\nabla}}$ are:

1. The gradient $\vec{\boldsymbol{\nabla}}$, e.g. $\vec{\boldsymbol{\nabla}}V(x,y,z) = -\vec{\boldsymbol{F}}(x,y,z)$, which converts a scalar field to a vector field.
2. The divergence, a dot product with $\vec{\boldsymbol{\nabla}}$, converts a vector field to a scalar field: e.g. $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}}(x,y,z) = f(x,y,z)$.
3. The curl, a cross product with $\vec{\boldsymbol{\nabla}}$ converts a vector field to another vector field: e.g. $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}(x,y,z) = \vec{\boldsymbol{G}}(x,y,z)$.

Vector and scalar fields can be integrated over a line in space, over a surface in space and over a volume in space using single, double or triple integrals. The detailed discussion of the multivariable calculus is outside the scope of this textbook, where we will only discuss the gradient and volume integrals over scalar fields.

4 Introduction to Dynamics

In this chapter we will introduce the field of dynamics, Newton's laws and the fundamental concepts of space, time, force and mass.

4.1 Dynamics and motion

Motion is all around you, the universe is full of moving matter and objects. This motion is surprisingly predictable and can therefore accurately be described by laws: Newton's laws of motion. The field of science and engineering studying the time-dependent motion of matter in the presence of forces is called *dynamics*. During several centuries dynamical models have been developed that enable scientists to understand many time-dependent processes in nature and have allowed engineers to use this understanding to design and control motion to an extremely high degree. Examples include precise robots, fast formula one racing cars and remotely controlled spacecrafts, all of which have been designed by engineers by applying the concepts that are introduced in this textbook. And who knows? Maybe you will design the next generation of dynamic machines and devices in the future.

Since the early history of mankind people have been studying motion. Astronomers in ancient times were observing the trajectories of celestial objects like moons and planets. Also, motion of objects on earth were being studied, like in the experiment in 1586 where Simon Stevin dropped two balls with different mass from the Nieuwe Kerk in Delft, providing evidence that they fall at the same rate. In 1687 Newton introduced the famous laws of motion in his Principia, which were based on the mathematics he had developed with Leibniz for solving differential and integral equations.

Newton's laws, which are the building blocks for the field of dynamics, provide a mathematical model to describe the experimentally observed motion of objects in our universe, which is accurate to a very high degree. Deviations from the model occur only in the case where speeds become comparable to the speed of light, such that Einstein's theory of relativity is needed or if sizes and energies and masses become very small, such that quantum mechanical effects start to play a role. These exceptions, which were only discovered in the 20^{th} century, belong to the field of modern physics and are outside the scope of this course, which focuses on classical mechanics and dynamics. In this textbook we will discuss the theoretical models of dynamics based on Newton's laws as if they represent a perfect description of experimental observations. Nevertheless, it remains important to realise that there is a difference between these models and the experimental observations they attempt to describe, since despite the high degree of accuracy of current physical models, there are still experiments that cannot be captured by them and are subject of thorough scientific investigations. For instance the model for describing the dynamics of stars rotating around the centre of a galaxy is still debated (gravity rotation problem) and the dynamics of particles that were observed to travel upstream into a small waterfall still remains to be explained. See also `https://en.wikipedia.org/wiki/List_of_unsolved_problems_in_physics`.

We note here that Newton's laws are not the only way to describe the experimentally observed motion of objects. Alternative popular formulations of dynamics are Lagrangian and Hamiltonian mechanics, which are dealt with in more advanced textbooks in dynamics. These formulations, yield completely identical results as Newton's laws, but can in some cases be advantageous to simplify the mathematical analysis.

4.2 Newton's laws of motion

In his Principia Newton introduced three laws of motion that can mathematically be expressed in vector format as follows:

Concept. *Newton's Laws of Motion*

$$1. \quad \text{If } \sum_j \vec{\boldsymbol{F}}_{ij} = \vec{\mathbf{0}} \text{ then } \vec{\boldsymbol{a}}_i = \vec{\mathbf{0}} \tag{4.1}$$

$$2. \quad \sum_j \vec{\boldsymbol{F}}_{ij} = m_i \vec{\boldsymbol{a}}_i \tag{4.2}$$

$$3. \quad \vec{\boldsymbol{F}}_{ij} = -\vec{\boldsymbol{F}}_{ji} \tag{4.3}$$

These equations and the variables will be explained and defined in more detail in the next two chapters. It can be seen that Newton's first law is a direct consequence of Newton's second law when setting $\sum \vec{\boldsymbol{F}}_{ij} = \vec{\mathbf{0}}$. Newton's third law (action=–reaction), describes a property of all known fundamental forces of nature, which states that for every force $\vec{\boldsymbol{F}}_{ij}$ that acts on point mass i there is *another* collinear force $\vec{\boldsymbol{F}}_{ji}$ that acts on *another* point mass j with equal magnitude and opposite direction. In other words if point mass j generates a force $\vec{\boldsymbol{F}}_{ij}$ on object i, then point mass i generates a force $\vec{\boldsymbol{F}}_{ji} = -\vec{\boldsymbol{F}}_{ij}$ on point mass j. The points of action of both forces coincide with the positions of the corresponding point masses. The first and third laws should be familiar if you followed a course in *statics*. Newton's second law $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ is the essential equation in *dynamics*.

Note. *Newton's laws are strictly only valid for objects that are point masses, which are so small that they can be described by a single point (position vector) in space. By applying Newton's laws to each point mass in an object or system that consists of many point masses, new laws can be derived for these larger systems. Much of this textbook is dedicated to deriving and discussing these new laws and using them to analyse more complex dynamical systems, like rigid bodies.*

4.3 Outline and elements of dynamics

Newton's laws are based on the following elemental concepts.

Concept. *Elements of dynamics*

- Space
  - Every location in space is identified by a position vector $\vec{\boldsymbol{r}}$.
- Time
  - Position vectors can change in time t, such that there is motion.
- Mass
  - Objects can consist of one or more point masses i, each having a well defined position vector $\vec{\boldsymbol{r}}_i$ and mass m_i.
- Force
  - On every point mass i one or more force vectors $\vec{\boldsymbol{F}}_{ij}(t)$ can act.

In the following chapters we will discuss these elemental concepts in more detail to describe the dynamic motion of objects, and then continue to discuss how this motion is related to *mass* and *force*. The subject of dynamics can be described as follows.

Concept. *Dynamics*
Dynamics is the field of science and engineering concerned with predicting and analysing the motion $\vec{\boldsymbol{r}}_i(t)$ of point masses under the influence of forces and constraints.

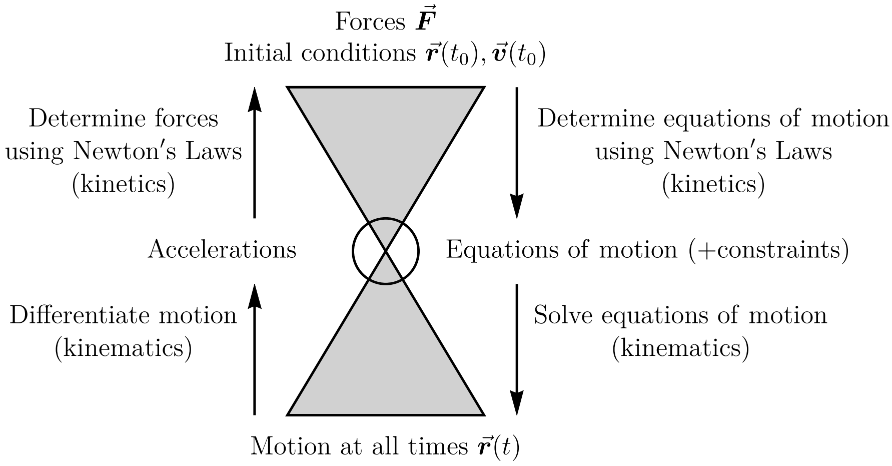


Figure 4.1: Determining the motion and forces in dynamics.

Since all matter in our universe consists of point masses, which are called elementary particles and move in response to four fundamental forces of nature, the field of dynamics covers practically all science with subdisciplines like classical dynamics, electrodynamics, quantum dynamics, and thermodynamics, but also chemistry and biology. In this textbook we will only discuss the dynamics that can be described by Newton's laws, which is generally called *classical dynamics*. The field of dynamics can be split into two main domains:

1. *Kinematics*
   - Study of the motion of objects in space and time without considering the forces that cause this motion.
2. *Kinetics*
   - Study of the motion of objects in relation to mass and forces.

Fig. 4.1 illustrates the roles of kinematics and kinetics in dynamics. Kinetics is used to relate the forces to the accelerations and equation of motion, while kinematics is used to relate the equations of motion (which are differential equations) and constraint equations to the actual motion. The figure illustrates that one can predict the motion $\vec{r}(t)$ at all times if the forces and initial conditions are known, and that it is also possible to determine the forces if the motion is known.

Besides the division into kinematics and kinetics, there is also a division of dynamics based on the types of objects that are being analysed:

1. *Point masses*
   Objects whose mass is located on a single point in space.
2. *Rigid bodies*
   Objects whose mass distribution in space is rigid and undeformable.
3. *Deformable bodies and fluids*
   Objects whose mass distribution in space is deformable like in elastic solids or can flow like a liquid or gas.

In part II we will discuss the dynamics of point masses, while in part III we will analyse the dynamics of rigid bodies. Deformable bodies are out of the scope of this textbook. For both parts we will discuss kinematics, kinetics, the method of work and energy and the method of impulse and momentum. Finally we will conclude with the discussion of vibrations. This division leads to the following outline of the rest of the chapters of this textbook:

- Kinematics of point masses: Ch. 5
- Kinetics of point masses: Ch. 6
- Work and energy of point masses: Ch. 7
- Impulse and momentum of point masses: Ch. 8
- Kinematics of rigid bodies: Ch. 9
- Kinetics of rigid bodies: Ch. 10
- Work and energy of rigid bodies: Ch. 11
- Impulse and momentum of point masses: Ch. 12
- Vibrations of point masses and rigid bodies: Ch. 13
- Solution strategy dynamics: Ch. 14
- Bibliography: Ch. 15

Part II

Dynamics of Point Masses

5 Kinematics of Point Masses

In this chapter we will show how to describe and analyse the dynamic motion of objects in space and time. We begin by defining what we mean by space, time, motion and kinematics in Sec. 5.1. Then we will describe what displacement, velocity and acceleration vectors are in Secs. 5.3–5.5. Subsequently we introduce the path coordinate, that allows simplifying motion in 3D to a 1D problem and show how to determine position, velocity and accelerations along the path curve in Secs. 5.7 and 5.8, and also show how to do this using various 3D coordinate systems (Sec. 5.9). Finally we discuss kinematic analysis of segmented motion in Sec. 5.10.

5.1 Space, time and motion

We start by introducing the concepts of space, time and motion, the building blocks of kinematics.

Concept. *Space*
In space the position of each point mass i can be described by a position vector $\vec{\boldsymbol{r}}_i$. Space can be parametrised by a three-dimensional (3D) coordinate system as introduced in Sec. 3.3. This space, in which the shortest distance between two points is a straight line, is also called Euclidean space.

Concept. *Time*
The position of objects in space can change[1] in time t.

Concept. *Motion*
The motion $\vec{\boldsymbol{r}}_i(t)$ of a single point-like object i is a function that describes its position coordinates at every time t (see Fig. 5.1). The motion of a system of N point masses is represented by a set of vector functions $\vec{\boldsymbol{r}}_1(t), \vec{\boldsymbol{r}}_2(t), \ldots, \vec{\boldsymbol{r}}_i(t), \ldots, \vec{\boldsymbol{r}}_N(t)$ that describe the positions of all points i in the system at all times.

[1] Note that without this change, it would be impossible to measure time, since devices to measure time like pendulum clocks are also based on dynamics and change.

Concept. *Kinematics*
Kinematics describes the motion of objects in time and space, irrespective of the forces that cause them to move. Kinematic techniques allow one to determine the relations between position, velocity and acceleration vectors, and to apply constraint equations for motion analysis.

Kinematic information on a point mass is provided in two ways:

1. Full or partial information on the time dependence of the position, velocity or acceleration vectors or its components.
2. Constraint equations that pose certain limitations on how the point mass can move.

The goal of kinematics is to analyse the given information to obtain more detailed information on the motion of the particle. The main challenges and skills in kinematics we will address in this chapter are:

1. To describe the motion of a point mass using a time-dependent position vector $\vec{\boldsymbol{r}}(t)$ in a suitable coordinate system.
2. To relate and determine position, velocity and acceleration vectors from each other.
3. To use constraint equations and equations of motion to determine and predict the time-dependent motion $\vec{\boldsymbol{r}}(t)$ of point masses.

5.2 Constrained and predetermined kinematics

In the next chapter on kinetics we will discuss how the motion of objects in time and space can be described based on the knowledge of forces and masses. However, often there is also different information about the motion available. For example when a train is constrained by a rail-track, when a measurement of the motion of a bird's trajectory was made, or when it is assumed that a rolling ball always touches a curved surface. This information can be cast in the form of constraint equations.

5.2.1 Constraint equations

Although all dynamics is governed by Newton's laws, rigorously applying these laws for every atom can be very inefficient. For instance, when a hockey puck is sliding on ice, modelling the forces of all ice molecules would be a tremendous task. Instead, we know that the ice molecules generate forces on the puck that prevent it from entering the ice, while gravity prevents the puck from

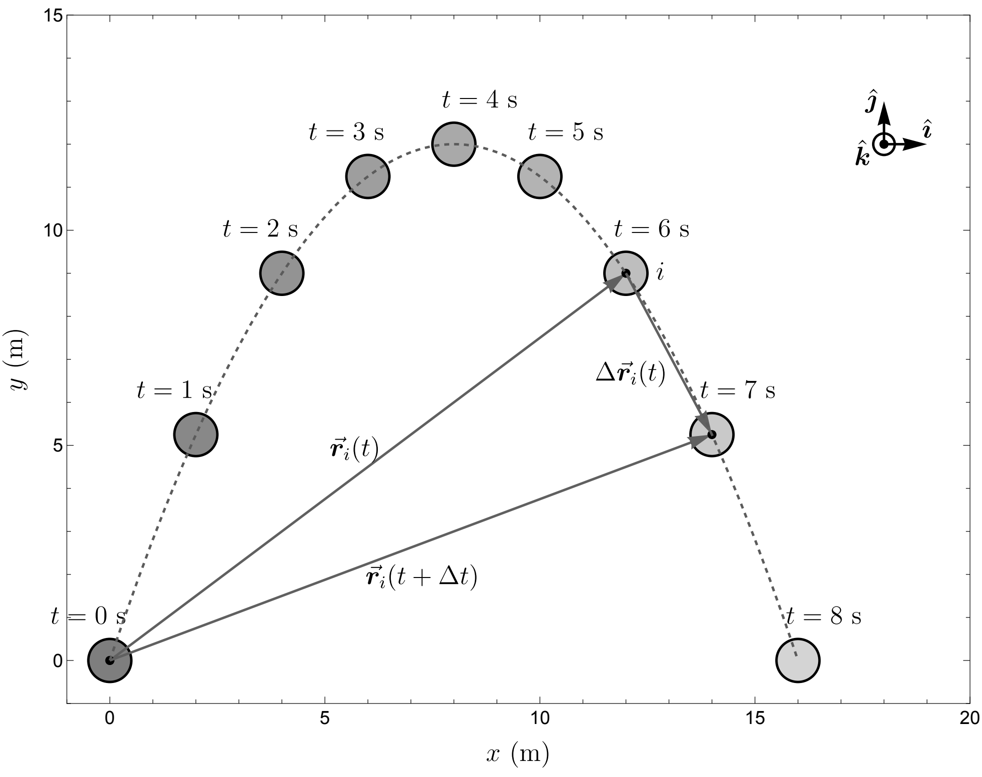


Figure 5.1: The motion of a point mass i is described by a time-dependent position vector function $\vec{\boldsymbol{r}}_i(t)$.

moving above the ice surface, such that the puck will obey a constraint equation $z(t) = 0$.

Thus in such cases, where we *know* the effective properties of a number of forces, it is much more efficient to describe their combined effect on the motion by *constraint equations* instead of applying Newton's laws. Remember however, that such constraint equations always are the result of forces, and the accuracy of the resulting dynamics will depend on the accuracy by which the constraint equations represent these forces. In the next chapter we will also discuss how constraint equations can be combined with Newton's laws to determine the forces generated by the constraints.

In other cases, information about the motion of an object has been obtained from a measurement, or is assumed to be known, and a constraint equation that describes the measured trajectory can be used to analyse the kinematics, and determine velocities and accelerations.

Definition. *Constraint equation*
Equation that provides information on the motion and trajectories of objects in time and space, without providing direct information on the forces and mass. Constraint equations describe the kinematic constraints, which are the constraints that limit the potential motion of a mechanical system.

A constraint equation f can often be written as a function of the position vectors of the objects $\vec{\boldsymbol{r}}_i$, and their time derivatives as:

$$f(\vec{\boldsymbol{r}}_i, \dot{\vec{\boldsymbol{r}}}_i, \ddot{\vec{\boldsymbol{r}}}_i, \ldots, t) = 0 \tag{5.1}$$

Where $\vec{\boldsymbol{r}}_i$, with $i = 1, 2, \ldots, N$ (or $i = A, B, \ldots$) are the position vectors of the objects in the system. By taking the time-derivative or time-integral of a constraint equation, additional constraint equations can be derived. We will now discuss several examples of constraint equations.

5.2.2 Spatially constrained kinematics

Although space is always three-dimensional, in some cases there are forces on the object that constrain the motion to a certain part of space. The analysis of the dynamics in these cases can be significantly simplified. For instance, when a hockey puck A slides over a flat ice surface, and we know or assume that it will not lift from the surface, this can be described by the constraint equation:

$$z_A(t) = 0 \tag{5.2}$$

Since in this case the puck will not move along the z_A coordinate, its dynamics can be fully dealt with in 2 dimensions (2D) using (x, y) coordinates or polar

(ρ, ϕ) coordinates, simply applying the Cartesian and cylindrical coordinate systems from the previous sections with $z = 0$. In fact this textbook deals with many examples of this 2D in-plane dynamics situation, the derived equations and techniques are valid in 3D unless explicitly stated otherwise.

5.2.3 Path curve and path coordinate

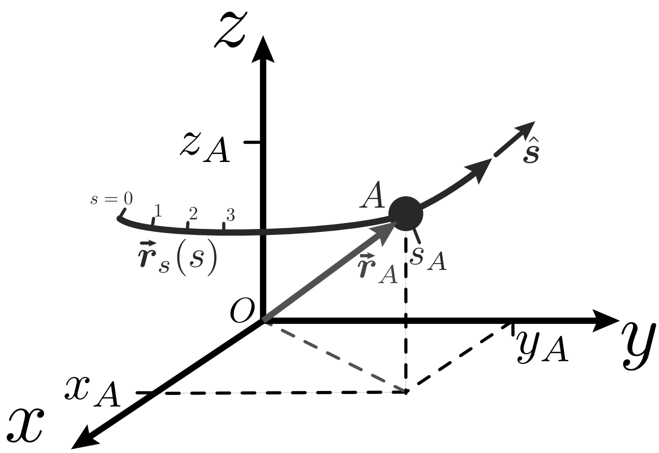


Figure 5.2: A mass A moves along path coordinate curve $\vec{r}_s(s)$ (blue). Its position, and distance from point $s = 0$ along the path curve, is uniquely determined by its scalar path coordinate s_A, with $\vec{r}_A = \vec{r}_s(s_A)$. The unit vector $\hat{s}$ points in the direction of increasing s. Note that the mass can move in the opposite direction if $\dot{s}_A < 0$.

An even more constrained situation is when the motion of an object is limited to a one-dimensional (1D) curvilinear path. This situation arises for instance when a train, marble or other point mass i moves along a predefined track, guided by a rail or tube.

As shown in by the blue curve in Fig. 5.2 such a curvilinear path traces out a 1D path in 3D space, and this path can be described by a parametrised *path curve* $\vec{r}_s(s)$, such that for every value of the scalar *path coordinate* s there is 1 vector $\vec{r}_s(s)$ that defines the position on the track. It is convenient (although not always easy), to choose this function such that s is a measure of the distance along the track with respect to a certain origin $\vec{r}_s(s = 0)$, for instance this can be done by taking a flexible tape measure, and measuring the distance s along the track. Since it is known that the point mass i always

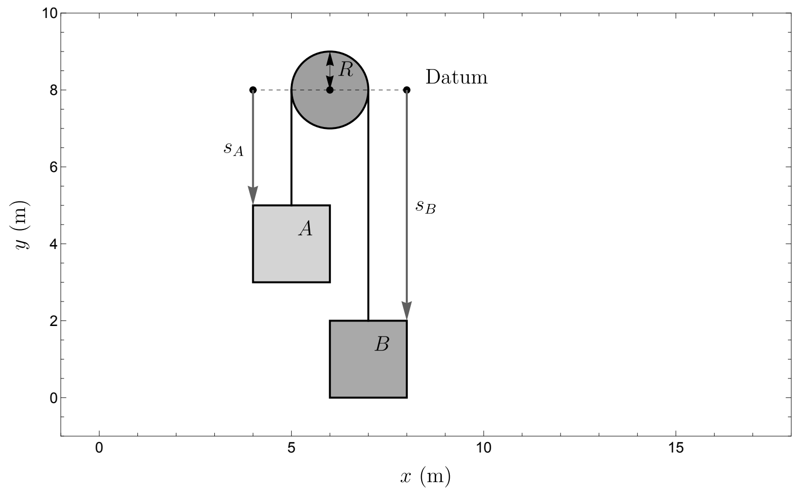


Figure 5.3: Two point masses A and B hang via a tight black rope over a pulley with radius R. They each have their own path coordinate s_A and s_B.

resides on the path curve, we have the following constraint equation:

$$\vec{\boldsymbol{r}}_i(t) = \vec{\boldsymbol{r}}_s(s). \tag{5.3}$$

At every time t, the path coordinate of object i has to obey equation (5.3), such that: $\vec{\boldsymbol{r}}_i(t) = \vec{\boldsymbol{r}}_s(s_i(t))$. This has the advantage that, because the path curve $\vec{\boldsymbol{r}}_s(s)$ is known, the motion of i in 3D can be described by a scalar function $s_i(t)$, instead of requiring a more complex 3D vector function $\vec{\boldsymbol{r}}_i(t)$. Equation (5.3) and also the equation $z_i(t) = 0$ from the previous subsection are examples of *constraint equations*.

5.2.4 Relatively constrained kinematics and pulleys

Besides spatial constraints that limit the motion of a point mass to a certain plane or curve in space discussed in the previous subsection 5.2, it is also possible to have constraints that limit motion of a point mass with respect to one or more other point masses. Consider for example two point masses A and B with their motion described by separate path curves and path coordinates $s_A(t)$ and $s_B(t)$, see Fig. 5.3. When making such a drawing, you are free to choose for each path coordinate the positive direction of the path curve and also the point at which the path coordinate is zero, which is called the *datum*. It is important to indicate those in the drawing.

We can write down a *relative constraint equation* for the length L_{rope} of the rope. From Fig. 5.3 we see that always:

$$s_A(t) + s_B(t) + R\pi = L_{\text{rope}} \tag{5.4}$$

The term $R\pi$ comes from the part of the rope that touches the pulley with radius R over half of its perimeter. By taking the time derivatives of this equation and using that the length of the rope L_{rope} and the radius of the pulley R is constant in time we find:

$$\begin{aligned} s_A(t) + s_B(t) &= c_1 = \text{constant} && (5.5)\\ \dot{s}_A(t) + \dot{s}_B(t) &= 0 && (5.6)\\ \ddot{s}_A(t) + \ddot{s}_B(t) &= 0 && (5.7) \end{aligned}$$

This shows that at all times the velocity and acceleration of the masses are equal and in opposite directions ($\ddot{s}_A = -\ddot{s}_B$). These relative constraint equations facilitate analysis of dynamic problems, since one does not need to determine the force in the rope. Since one is usually only interested in the constraint equation for the velocities, it is often not necessary to determine the exact value of the constant $c_1 = L_{\text{rope}} - R\pi$.

The shown procedure for obtaining the constraint equations is also quite generally applicable:

- Determine the path coordinates of all moving objects and point masses based on the geometry.
- Determine the constraint equations for the positions of the objects and simplify them.
- Take two times the time derivative of the constraint equation to obtain the constraint equations for the velocities, and for the accelerations.

S **Example 5.1** Fig. 5.4 shows a more complicated pulley system. The question is: find the relative constraint equation that relates the velocities $\dot{s}_A$, $\dot{s}_B$ of point masses A and B.

Exemplary solution Since pulleys C and D are also moving, it is useful to also monitor their path coordinates s_C and s_D, which have been indicated in Fig. 5.4. Now we write down all constraint equations and combine them to obtain a single equation relating s_A and s_B:

$$\begin{aligned} L_{\text{rope}} &= s_A + 2s_C + s_D + L_{CD} + c_1 && (5.8)\\ s_C &= s_B - (L_{CD} + L_B) && (5.9)\\ s_D &= s_B - L_B && (5.10)\\ L_{\text{rope}} &= s_A + 3s_B + c_2 && (5.11) \end{aligned}$$

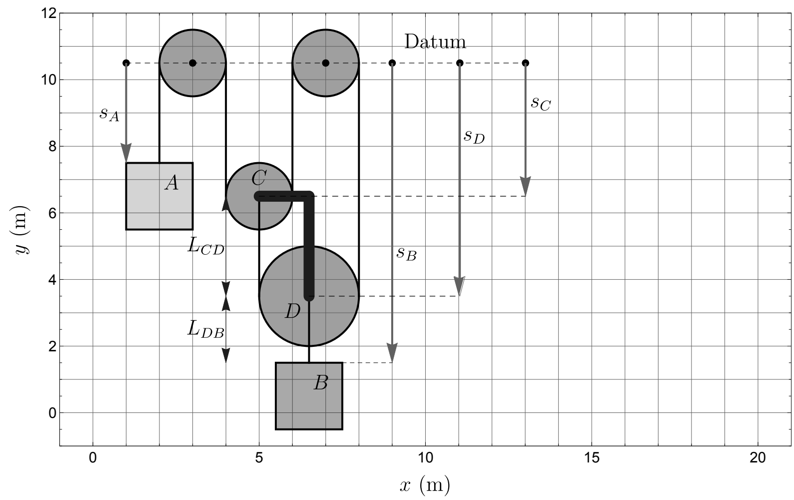


Figure 5.4: Example 5.1: Two point masses A and B move in a complex pulley system. We want to find their relative constraint equation. Pulleys C and D are connected by a rigid blue rod to keep their mutual distance at L_{CD}.

The actual value of the constant c_2 is not needed to solve the problem, since we now take the time derivative to obtain the relative constraint equation for the velocities:

$$s_A + 3s_B = c_3 = \text{constant} \tag{5.12}$$

$$\dot{s}_A + 3\dot{s}_B = 0 \tag{5.13}$$

$$\dot{s}_A = -3\dot{s}_B \tag{5.14}$$

The three constraint equations (5.8-5.10) represent the effect of the three constraining elements in Fig. 5.4: the rope, the rigid rod between pulleys C and D and the short rope between pulley D and point mass B. Initially there are four unknown scalar variables: s_A, s_B, s_C and s_D. Generally each (scalar) constraint equation eliminates one unknown, such that in the end we are left with one constraint equation (with one unknown). Besides pulleys and ropes there are of course many other elements that constrain motion, which will be discussed later.

5.3 Displacement

After having discussed how the position of a point mass i can be described in space using coordinate systems and vectors, we now turn to how its position evolves in time, and introduce the important concepts of *displacement, velocity* *and acceleration* that are essential for analysing dynamics. For this purpose we compare the position $\vec{\boldsymbol{r}}_i(t_1)$ at a first time t_1 to $\vec{\boldsymbol{r}}_i(t_2)$ at a later time $t_2 = t_1 + \Delta t$, with $\Delta t > 0$. The difference between these vectors, *the change in position vector*, is the displacement vector as can be seen in Fig. 5.1.

Concept. *Displacement vector*
The displacement vector $\Delta\vec{\boldsymbol{r}}_{i,12}$ describes the displacement in space of a point mass between times t_1 and t_2.

$$\Delta\vec{\boldsymbol{r}}_{i,12} = \vec{\boldsymbol{r}}_i(t_2) - \vec{\boldsymbol{r}}_i(t_1). \tag{5.15}$$

The initial and final position are separated by a *displaced distance* $\Delta r_{i,12} = |\Delta\vec{\boldsymbol{r}}_{i,12}|$. The *displaced path distance* $\Delta s_{i,12}$ of the mass, which is the displaced distance as measured along the path (instead of along a straight line) can be determined by the equation:

$$\Delta s_{i,12} = s_i(t_2) - s_i(t_1), \tag{5.16}$$

when the trajectory of the mass is parametrised by a path curve as discussed in the previous section 5.2.3.

The total travelled distance $\Delta s_{\mathrm{T},i,12}$ is not the distance between the initial and final positions, but depends on the path taken. The travelled distance can be larger than the displaced distance $\Delta s_{i,12}|$ if the mass reverses direction on its path and travels back and forth along the same path curve in multiple segments, in which case the distances $|\Delta s_{i,12}|$ along each of these segments (with different sign $\Delta s_{i,nm}$) should be summed up to determine $\Delta s_{\mathrm{T},i,12}$.

If the path is a straight line, the displaced path distance $|\Delta s_{i,12}|$ is equal to the displaced distance $|\Delta\vec{\boldsymbol{r}}_{i,12}|$, but when the path is curved, the displaced path distance is always longer, because a straight line is the shortest distance between two points: $|\Delta s_{i,12}| \geq |\Delta\vec{\boldsymbol{r}}_{i,12}|$.

The *infinitesimal displacement vector* that corresponds to an infinitesimally small time difference $\mathrm{d}t$ with $t_1 = t$ and $t_2 = t + \mathrm{d}t$ is found to be:

$$\mathrm{d}\vec{\boldsymbol{r}}_i = \vec{\boldsymbol{r}}_i(t + \mathrm{d}t) - \vec{\boldsymbol{r}}_i(t). \tag{5.17}$$

The closer you look at a small segment of a smooth trajectory, the more it resembles a straight line. This can be seen in Fig. 5.1 by comparing the dashed line and the straight arrow between $\vec{\boldsymbol{r}}_i(6\mathrm{s})$ and $\vec{\boldsymbol{r}}_i(7\mathrm{s})$. Therefore the infinitesimally travelled distance $\mathrm{d}s_i = s_i(t + \mathrm{d}t) - s_i(t)$ becomes equal to the length of the infinitesimal displacement vector: $|\mathrm{d}s_i| = |\mathrm{d}\vec{\boldsymbol{r}}_i|$.

5.4 Velocity

Velocity is displacement per unit of time. The *average velocity* of point mass i between time t_1 and t_2, over path curve $\vec{\boldsymbol{r}}_s$, over a time interval $\Delta t = t_2 - t_1$ is:

$$v_{s,\text{avg}} = \frac{\Delta s_{i,12}}{\Delta t} \tag{5.18}$$

This average velocity can be positive or negative. In contrast, the *average speed*, is defined to be always positive:

$$v_{s,\text{sp,avg}} = \frac{\Delta s_{\text{T},i,12}}{\Delta t} \tag{5.19}$$

The average velocity and speed are determined over a large time interval, in contrast the instantaneous velocity is measured over an infinitesimally small time interval.

Concept. *Velocity vector*
The (instantaneous) velocity vector $\vec{\boldsymbol{v}}_i$ is defined as the time derivative of the position vector:

$$\vec{\boldsymbol{v}}_i(t) \equiv \frac{\mathrm{d}\vec{\boldsymbol{r}}_i}{\mathrm{d}t} \equiv \lim_{\mathrm{d}t \to 0} \frac{\vec{\boldsymbol{r}}_i(t + \mathrm{d}t) - \vec{\boldsymbol{r}}_i(t)}{\mathrm{d}t}. \tag{5.20}$$

The determination of the velocity vector from the position vector is shown graphically in Fig. 5.5, taking a relatively large timestep $\mathrm{d}t = 1\,\mathrm{s}$. The *speed or absolute velocity* $v_i(t)$ is defined as the magnitude of the velocity vector, $v_i(t) = |\vec{\boldsymbol{v}}_i(t)|$ which is always positive or zero.

Derivation. *Velocity on the path curve*
If a path curve has been defined, it can be shown that the velocity vector is always tangential to this path curve:

$$\vec{\boldsymbol{v}}_i = \frac{\mathrm{d}\vec{\boldsymbol{r}}_i}{\mathrm{d}t} = \frac{\mathrm{d}\vec{\boldsymbol{r}}_s(s_i(t))}{\mathrm{d}t} = \frac{\mathrm{d}s_i}{\mathrm{d}t}\frac{\mathrm{d}\vec{\boldsymbol{r}}_s(s)}{\mathrm{d}s} = v_{s,i}\hat{\boldsymbol{s}}. \tag{5.21}$$

To derive this equation, we used the chain rule and used that the vector $\frac{\mathrm{d}\vec{\boldsymbol{r}}_s(s)}{\mathrm{d}s}$ has a magnitude of 1, since $|\mathrm{d}s| = |\mathrm{d}\vec{\boldsymbol{r}}|$ (see previous section). As can be seen in Eq. (5.21), we define $\hat{\boldsymbol{s}} \equiv \frac{\mathrm{d}\vec{\boldsymbol{r}}_s(s)}{\mathrm{d}s}$ as the unit vector tangential to the path curve that points in the direction of the path curve in which s increases (see Fig. 5.2) and we define the path velocity v_s as:

$$v_{s,i} \equiv \frac{\mathrm{d}s_i}{\mathrm{d}t} = \dot{s}_i. \tag{5.22}$$

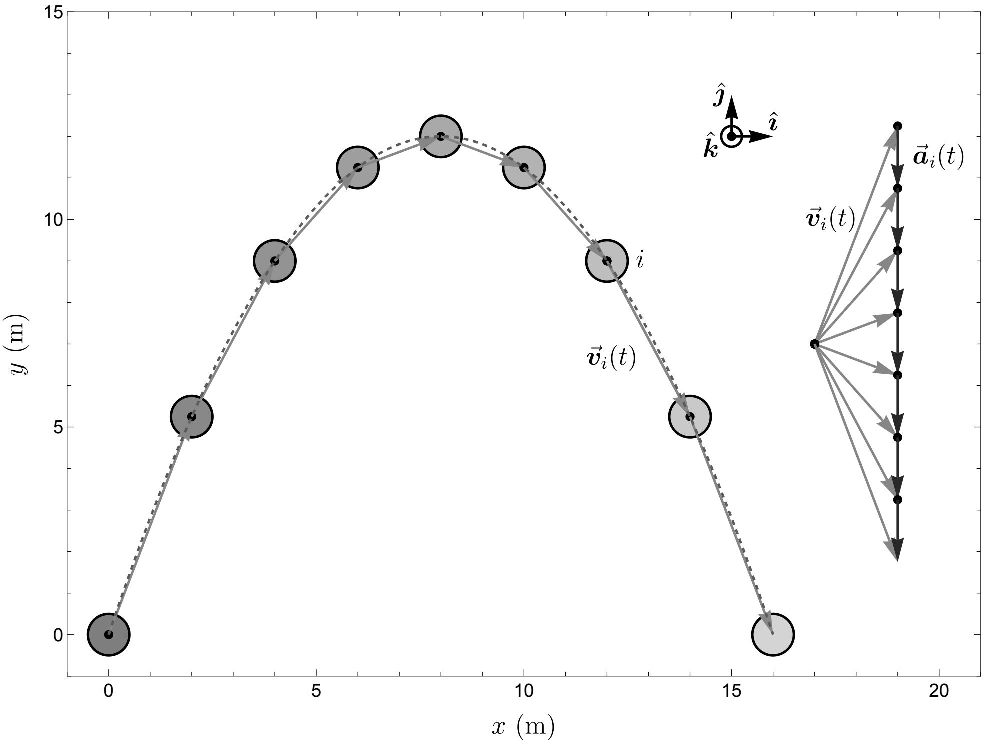


Figure 5.5: Based on the motion of the point mass in Fig. 5.1, the velocity vectors $\vec{v}_i(t)$ are determined at every time t, using a time-step $\mathrm{d}t = 1\,\mathrm{s}$. On the right the acceleration vectors $\vec{a}_i(t)$ are determined from $\vec{v}_i(t)$ in the same way as the velocity vectors were determined from the position vectors. Note that the average velocity vectors between the points with $\Delta t = 1$ s are drawn.

The subscript s indicates that $v_{s,i}$ is the velocity of object i *along the path* $\vec{\boldsymbol{r}}_s$. Equation (5.21) proves that the velocity vector of the point mass i is always parallel (tangential) to the path curve unit vector $\hat{\boldsymbol{s}}$. $v_{s,i}$ can be both positive and negative, depending whether it is in the same or the opposite direction as the positive s direction $\hat{\boldsymbol{s}}$ and its magnitude is equal to that of the velocity vector $v_i \equiv |\vec{\boldsymbol{v}}_i| = |v_{s,i}|$.

5.5 Acceleration

Similarly to the way the velocity vector is determined from the position vector, the acceleration vector can be determined from the velocity vector (see Fig. 5.5).

Concept. *Acceleration vector*
The (instantaneous) acceleration vector $\vec{\boldsymbol{a}}_i$ *is defined as the time derivative of the velocity vector:*

$$\vec{\boldsymbol{a}}_i(t) \equiv \frac{\mathrm{d}\vec{\boldsymbol{v}}_i}{\mathrm{d}t} \equiv \lim_{\mathrm{d}t \to 0} \frac{\vec{\boldsymbol{v}}_i(t+\mathrm{d}t) - \vec{\boldsymbol{v}}_i(t)}{\mathrm{d}t}. \tag{5.23}$$

The *absolute acceleration* is defined as: $a_i = |\vec{\boldsymbol{a}}_i|$. By taking the time derivative of equation (5.21) using the product rule it is found that:

$$\vec{\boldsymbol{a}}_i(t) = \frac{\mathrm{d}\vec{\boldsymbol{v}}_i}{\mathrm{d}t} = \frac{\mathrm{d}v_{s,i}}{\mathrm{d}t}\hat{\boldsymbol{s}} + v_{s,i}\frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t} \tag{5.24}$$

In contrast to the velocity vector, the acceleration vector is not tangential to the path curve $\hat{\boldsymbol{s}}$, except if the path is straight ($\frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t} = \vec{\boldsymbol{0}}$), or if the speed is zero. Since the magnitude of $\hat{\boldsymbol{s}}$ is constant, the vector $\frac{\mathrm{d}\hat{\boldsymbol{s}}}{\mathrm{d}t}$ is perpendicular to $\hat{\boldsymbol{s}}$. For that reason we have for the acceleration component tangential to the path curve $a_{i,s,t}$:

$$a_{s,i,t} = \vec{\boldsymbol{a}}_i \cdot \hat{\boldsymbol{s}} = \frac{\mathrm{d}v_{s,i}}{\mathrm{d}t} \tag{5.25}$$

This tangential component of the acceleration vector is also called the path acceleration $a_{s,i}$ and can be different from the absolute value of the acceleration vector ($|a_{s,i}| \neq |\vec{\boldsymbol{a}}_i|$). We will discuss later how the acceleration vector in various coordinate systems can be determined, which will be very important for applying Newton's second law.

5.6 The equations of motion

When analysing dynamic systems we often want to determine the motion $\vec{\boldsymbol{r}}(t)$ from a known acceleration $\vec{\boldsymbol{a}}(t)$, velocity $\vec{\boldsymbol{v}}(t)$ or a more complicated differential equation that involves the position coordinates and their time-derivatives. Such an equation is called an equation of motion (EoM). Often multiple equations are needed to fully determine the time-dependent motion $\vec{\boldsymbol{r}}_i(t)$ of every point in the system.

Concept. *Equations of motion*
The equations of motion of a system are a set of differential equations of the position vectors/coordinates and their time-derivatives, that can be used to determine the motion $\vec{\boldsymbol{r}}_i(t)$ of every point i in the system for any initial condition.

We also introduce the concepts initial condition and state:

Concept. *Initial condition and state of a system*
Initial conditions fully describe the state of a system at a single initial time t_0. The state of the system is defined as the combination of the position $\vec{\boldsymbol{r}}_i(t_0)$ and velocity vectors $\vec{\boldsymbol{v}}_i(t_0)$ of all points i in the system at a single time t_0.

If the initial conditions, equations of motion and constraint equations are known, the future and history of a system can be calculated by kinematic techniques. As an example of an equation of motion let us consider the situation where we know that the velocity along a path curve is given by the function $v(t) = c_1 t^2$. Then from the definition of velocity and Eq. (5.22) we find the following equation of motion:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = c_1 t^2 \tag{5.26}$$

Such an equation of motion fixes the slope of the motion $s(t)$ at every (s, t) coordinate as is graphically shown by the arrows in Fig. 5.6. A main challenge in kinematic analysis is to determine the motion $s(t)$ from the EoMs and initial conditions as illustrated by the red lines in the figure. In this chapter we will discuss EoMs that can be solved by integration. In Ch. 13 on vibrations we will also deal with other solution methods.

5.7 Kinematic analysis as a function of time t

We will show how the motion, velocities and accelerations of a system can be determined by solving equations of motion using time differentiation and integration.

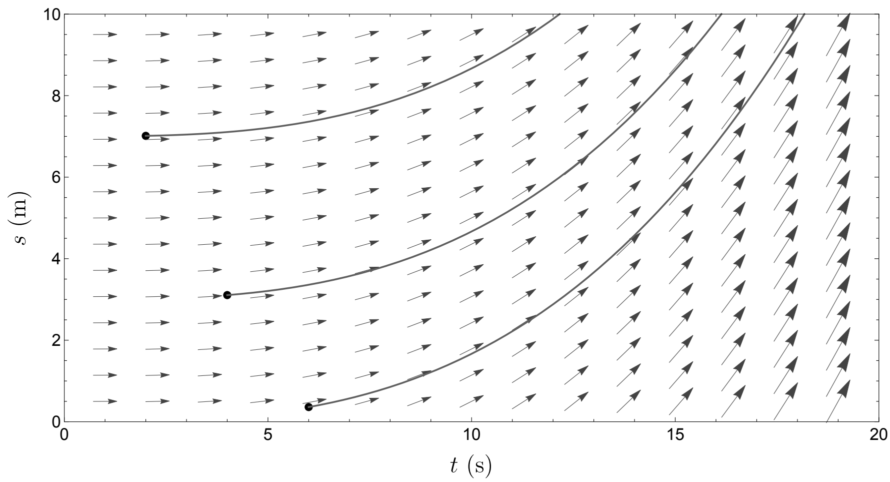


Figure 5.6: Graphical representation of the equation of motion $\frac{\mathrm{d}s}{\mathrm{d}t} = c_1 t^2$. The arrows indicate the slope $\frac{\mathrm{d}s}{\mathrm{d}t}$ at every value of s and t. For several initial conditions (black dots) the solutions of this EoM are plotted as red lines. It can be seen that the lines are everywhere tangent (have equal slope) to the arrows of the EoM.

5.7.1 Time differentiation

We have already shown how the velocity and acceleration vectors can be determined from the position vector by two time differentiation steps that are indicated by arrows:

$$\vec{\boldsymbol{r}}_i \quad \rightarrow \tag{5.27}$$
$$\vec{\boldsymbol{v}}_i \quad = \quad \dot{\vec{\boldsymbol{r}}}_i \rightarrow \tag{5.28}$$
$$\vec{\boldsymbol{a}}_i \quad = \quad \dot{\vec{\boldsymbol{v}}}_i \tag{5.29}$$

And the same can be done along the path coordinate s, simplifying the problem to scalar differentiation:

$$s_i(t) \quad \rightarrow \tag{5.30}$$
$$v_{s,i}(t) \quad = \quad \dot{s}_i(t) \rightarrow \tag{5.31}$$
$$a_{s,i}(t) \quad = \quad \dot{v}_{s,i}(t) = \ddot{s}_i(t) \tag{5.32}$$

During each of these steps, an ordinary differential equation (ODE) is solved, which is of the form of Eq. (2.12), $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = y(x)$, where x is replaced by t, $f(x)$ is replaced by the known function $s(t)$ or $v_s(t)$ and $y(x)$ is replaced by the unknown function $v_s(t)$ or $a_s(t)$ that we would like to determine. The details of the procedure to solve these first order differential equations using either indefinite or definite integrals is explained in Sec. 2.4.

5.7.2 Time integration

To go in the opposite way, from acceleration to velocity and position, an *equation of motion* needs to be solved to obtain $s_i(t)$ from $a_{s,i}(t)$, which is of the form:

$$\ddot{s}_i = a_{s,i}(t) \tag{5.33}$$

Because the function $a_{s,i}(t)$ only depends on time, this differential equation can be solved by integrating twice over time. Here we show how that is done by integration along the path curve, and in Eq. (5.57) it will be shown how it can be done in free space. Although these integrals can become difficult analytically, they can always be performed using numerical methods as will be discussed concisely in section 5.8.5.

If the path acceleration function $a_{s,i}(t)$ is known, it is possible to determine the path velocity and path position by integration of Eqs. (5.30)–(5.32) as follows.

$$a_{s,i}(t) \quad \rightarrow \tag{5.34}$$

$$v_{s,i}(t_2) = v_{s,i}(t_1) + \int_{t_1}^{t_2} a_{s,i}(t)\mathrm{d}t \rightarrow \tag{5.35}$$

$$s_i(t_2) = s_i(t_1) + \int_{t_1}^{t_2} v_{s,i}(t)\mathrm{d}t \tag{5.36}$$

During each of the integration steps, an ODE is solved of the form of Eq. (2.13). From equations like Eq. (5.35) it is noted that time integration to obtain final velocity and position can only be performed if the position $s_i(t_1)$ and velocity $v_{s,i}(t_1)$ are known at a certain instant t_1. That is why one needs to known the initial conditions to solve the equations of motion.

S **Example 5.2** To illustrate the kinematic analysis in time, let's consider the motion of a car along a track (Fig. 5.7), with path curve $\vec{\boldsymbol{r}}_s(s) = s\hat{\boldsymbol{\imath}}$ and path coordinate $s_A = x_A$. It is given that the position of the car A on the track is:

$$s_A(t) = \alpha t^3 \tag{5.37}$$

Then we can obtain the velocity and acceleration by differentiation using Eqs. (5.30)–

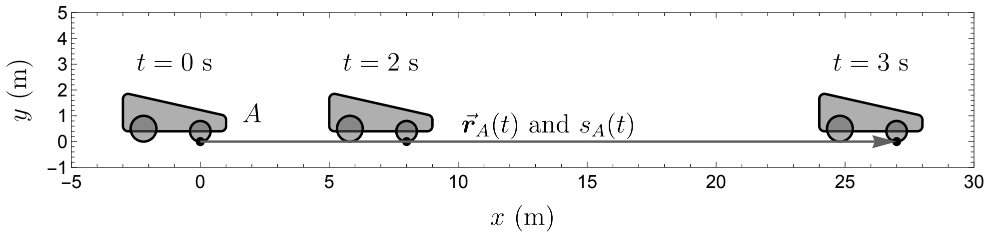


Figure 5.7: Example 5.2: positions of a red car with position $s_A(t) = \alpha t^3$ at three times, for $\alpha = 1\,\text{m/s}^3$. The car moves along a path curve $\vec{\boldsymbol{r}}_s(s_A) = s_A\hat{\boldsymbol{\imath}}$. Since the car is approximated by a point mass, only the motion of the point indicated by the black dot is analysed.

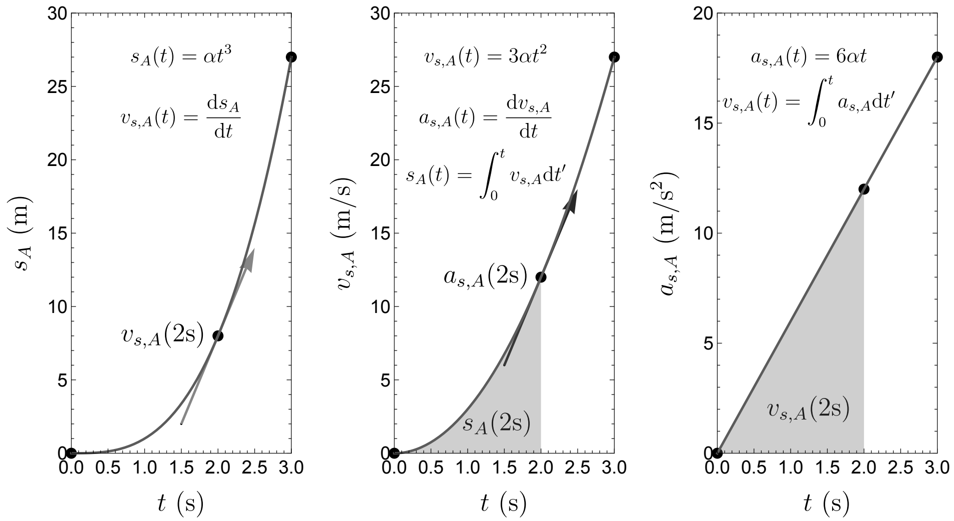


Figure 5.8: Example 5.2: Relations between the path coordinate s_A, path velocity $v_{s,A}$ and path acceleration $a_{s,A}$ of the car shown in Fig. 5.7 with $\alpha = 1\,\text{m/s}^3$. The path velocity $v_{s,A}(t)$ is the slope of the $s_A(t)$ curve and the path acceleration is the slope of the $v_{s,A}(t)$ curve as indicated by the tangential arrows. The path velocity $v_{s,A}(t)$ increase compared to $v_{s,A}(0)$ is the integral of the area under the $a_{s,A}(t)$ curve and the path coordinate $s_A(t)$ increase is the integral of the area under the $v_{s,A}(t)$ curve as indicated by the red areas.

(5.32):

$$v_{s,A}(t) = \dot{s}_A = \alpha\frac{\mathrm{d}}{\mathrm{d}t}t^3 = 3\alpha t^2 \quad (5.38)$$

$$a_{s,A}(t) = \dot{v}_{s,A} = \alpha\frac{\mathrm{d}}{\mathrm{d}t}3t^2 = 6\alpha t \quad (5.39)$$

So the velocity is the slope (=time-derivative) of the $s_A(t)$ graph and the acceleration is the slope of the $v_{s,A}(t)$ graph, as is shown by the arrows in Fig. 5.8.

Suppose now that instead, the acceleration of the car $a_{s,A}(t) = 6\alpha t$ is given and we want to determine the distance travelled after a certain time. It is given that the car starts at $t_1 = 0$ from rest, such that initial velocity and position of the car are both zero ($v_{s,A}(t_1) = s_A(t_1) = 0$). From the acceleration the functions that describe velocity and position at an arbitrary time t_2 can be obtained using equations (5.34)–(5.36):

$$a_{s,A}(t) = 6\alpha t \quad (5.40)$$

$$v_{s,A}(t_2) = v_{s,A}(t_1) + \int_{t_1}^{t_2} 6\alpha t\mathrm{d}t = 0 + \left[3\alpha t^2\right]_{t_1}^{t_2} = 3\alpha t_2^2 \quad (5.41)$$

$$s_A(t_2) = s_A(t_1) + \int_{t_1}^{t_2} 3\alpha t^2\mathrm{d}t = 0 + \left[\alpha t^3\right]_{t_1}^{t_2} = \alpha t_2^3. \quad (5.42)$$

So the velocity change is the area (=time-integral) under the $a_{s,A}(t)$ graph and the position change is the area (=time-integral) under the $v_{s,A}(t)$ graph, as is shown by the red areas in Fig. 5.8.

Comparing the first 3 equations of this example to the last 3 equations demonstrates that the time differentiation and integration results in the same equations for position, velocity and acceleration, as expected.

5.8 Kinematic analysis with the path coordinate s

In some cases the velocity $v_{s,i}(s)$, acceleration $a_{s,i}(s)$ or time $t_s(s)$ as a function of path coordinate s is known, but the explicit time dependence is unknown. In those cases it can be more convenient to choose s as an independent differentiation or integration variable than t.

Note well, the function $v_{s,i}(s)$ is mathematically different from the function $v_{s,i}(t)$, and that also holds for and $a_{s,i}(s)$ and $a_{s,i}(t)$. These functions are connected by the relations $v_{s,i}(s(t)) = v_{s,i}(t)$ and $a_{s,i}(s(t)) = a_{s,i}(t)$ that should hold at any time. The difference by these functions will be designated by clearly indicating between brackets the variable on which they depend ($v_{s,i}(t)$ or $v_{s,i}(s)$).

5.8.1 Path differentiation

If we take the time derivative of the function $v_{s,i}(s)$ we obtain using the chain rule:

$$a_{s,i}(s) = \frac{\mathrm{d}v_{s,i}(s)}{\mathrm{d}t} = \frac{\mathrm{d}v_{s,i}(s)}{\mathrm{d}s}\frac{\mathrm{d}s_i}{\mathrm{d}t} = \frac{\mathrm{d}v_{s,i}(s)}{\mathrm{d}s}v_{s,i}(s) \tag{5.43}$$

This equation can be used to obtain the acceleration function $a_{s,i}(s)$ from $v_{s,i}(s)$.

5.8.2 Path integration

When the acceleration $a_{s,i}(s)$ is known as a function of the path coordinate s instead of as a function of time, the following differential equation needs to be solved to obtain the velocity $v_{s,i}(s)$:

$$a_{s,i}(s) = \frac{\mathrm{d}v_{s,i}}{\mathrm{d}t} = \ddot{s}_i \tag{5.44}$$

After multiplying the equation on both sides by $\mathrm{d}s$, and using that $\frac{\mathrm{d}v_{s,i}}{\mathrm{d}t}\mathrm{d}s = \mathrm{d}v\frac{\mathrm{d}s_i}{\mathrm{d}t} = v_{s,i}\mathrm{d}v_{s,i}$ the differential equation can be solved by integration:

$$\int_{s_1}^{s_2} a_{s,i}\mathrm{d}s = \int_{v_1}^{v_2} v_{s,i}\mathrm{d}v_{s,i} = \left[\frac{1}{2}v_{s,i}^2\right]_{v_1}^{v_2} = \frac{1}{2}(v_2^2 - v_1^2) \tag{5.45}$$

Where v_1 and v_2 are the velocity of the point mass at the positions s_1 and s_2, $v_1 = v_{s,i}(s_1)$ and $v_2 = v_{s,i}(s_2)$. This equation can be rewritten in a form that allows finding $v_{s,i}$ from $a_{s,i}$:

$$v_{s,i}^2(s_2) = v_{s,i}^2(s_1) + 2\int_{s_1}^{s_2} a_{s,i}(s)\mathrm{d}s \tag{5.46}$$

We note that the relation $a_s\mathrm{d}s = v_s\mathrm{d}v_s$ that appears under the integrals in Eq. (5.45) is often useful to simplify expressions for differentiation and integration with the path coordinate s.

5.8.3 Velocity dependent acceleration

A final case that can become of relevance, is when acceleration $a_{s,i}(v)$ depends on velocity. To obtain the velocity as a function of time $v_{s,i}(t)$ one needs to solve this differential equation:

$$\frac{\mathrm{d}v_{s,i}}{\mathrm{d}t} = a_{s,i}(v) \tag{5.47}$$

The strategy for solving this kind of ODE is to rearrange all terms such that terms depending on t are on one side of the equal side and terms depending on v are on the other side. Here that is done by multiplying both sides by $\mathrm{d}t$ and dividing them by $a_{s,i}$ and then integrating:

$$\int_{v_1}^{v_s} \frac{\mathrm{d}v_{s,i}}{a_{s,i}(v)} = \int_{t_1}^{t_s} \mathrm{d}t = t_s - t_1 \tag{5.48}$$

By evaluating this integral and using the information on the initial condition at t_1, one obtains a function $t_s(v_s)$, which can be inverted to give the velocity as a function of time $v_s(t_s)$. Alternatively, if one is looking for the function $v_s(s)$ it is possible to multiply both sides of the differential equation (5.47) by $\mathrm{d}s$, divide by $a_{s,i}$ and integrate to obtain:

$$\int_{v_1}^{v_s} \frac{\mathrm{d}v_{s,i}}{a_{s,i}(v)} v_{s,i} = \int_{s_1}^{s} \mathrm{d}s' = s - s_1 \tag{5.49}$$

Using the initial condition this gives $s(v_s)$ which can be inverted to obtain $v_s(s)$.

5.8.4 Overview integration and differentiation in kinematics

After having discussed several ways to differentiate and integrate the kinematics when motion is along a one-dimensional path curve, we provide here an overview table with all equations for reference. The equations for integrating and differentiating between the different kinematic variables a, v, s and t as a function of each other are summarised in Table 5.1 below. Please note that the cases with t as dependent variable are usually the most important. Moreover, realise that it is better to learn how to properly derive these equations yourself by integrating and differentiation than to memorise equations.

5.8.5 Numerical differentiation and integration

Often, expressions are too complicated to differentiate or integrate using analytical equations. However in those cases the integration can usually still be performed numerically.

- Advantages of numerical methods:
  - Can solve almost all problems
  - No need to perform (difficult) analytical integration or differentiation
- Disadvantages of numerical methods:

	$a_s()$	$v_s()$	$s()$	$t_s()$
t	$a_s(t) = \frac{\mathrm{d}v_s(t)}{\mathrm{d}t}$	$v_s(t) = \frac{\mathrm{d}s(t)}{\mathrm{d}t}$	$s(t) = t_s^{\langle -1\rangle}(s)$	
t		$v_s(t) = v_s(t_1) + \int_{t_1}^{t} a_s(t')\mathrm{d}t'$	$s(t) = s(t_1) + \int_{t_1}^{t} v_s(t')\mathrm{d}t'$	
s	$a_s(s) = v_s(s)\frac{\mathrm{d}v_s(s)}{\mathrm{d}s}$	$v_s(s) = \left(\frac{\mathrm{d}t_s(s)}{\mathrm{d}s}\right)^{-1}$		$t_s(s) = s^{\langle -1\rangle}(t)$
s		$v_s^2(s) = v_s^2(s_1) + 2\int_{s_1}^{s} a_s(s')\mathrm{d}s'$		$t_s(s) = t_s(s_1) + \int_{s_1}^{s} \frac{\mathrm{d}s'}{v_s(s')}$
v	$a_s(v) = \left(\frac{\mathrm{d}t_s(v)}{\mathrm{d}v}\right)^{-1}$		$s(v) = v_s^{\langle -1\rangle}(s)$	$t_s(v) = v_s^{\langle -1\rangle}(t)$
v	$a_s(v) = v\left(\frac{\mathrm{d}s(v)}{\mathrm{d}v}\right)^{-1}$		$s(v) = s(v_1) + \int_{v_1}^{v} \frac{v'\mathrm{d}v'}{a_s(v')}$	$t_s(v) = t_s(v_1) + \int_{v_1}^{v} \frac{\mathrm{d}v'}{a_s(v')}$

Table 5.1: Equations to relate different kinematic quantities along the path curve a, v, s and t as a function of each other. The top row indicates the function and dependent variable, while the left column indicates the independent variable. For example, the function $s(t)$ can be obtained in 2 ways: either by time integrating the function $v_s(t)$ or by taking the inverse function of the function $t_s(s)$. Note that $t_s^{\langle -1\rangle}(s)$ in this table indicates the inverse function of $t_s(s)$, not $1/t_s(s)$.

- Less accurate and fast than computing an analytic solution
- Programming required
- Gives less insight because only one specific solution is found, not the dependence on variables

To numerically determine the time derivative of a known function $s_A(t)$, you determine the position s_A at two closely spaced points in time $s_A(t)$ and $s_A(t + \Delta t)$, with a small time difference Δt. The simplest numerical differentiation in formula form is then performed using this equation:

$$v_{s,A}(t) \approx \frac{s_A(t + \Delta t) - s_A(t)}{\Delta t} \tag{5.50}$$

This equation can be applied for all values of t to obtain $v_{s,A}(t)$. According to the definition of the derivative, this expression becomes exactly correct in the limit $\Delta t \to 0$ which allows improving the numerical accuracy as much as needed by reducing the time step Δt.

If instead the velocity $v_{s,A}(t)$ is known, the position s_A can be determined by numerical time integration by just rewriting the previous equation to:

$$s_A(t + \Delta t) = s_A(t) + v_{s,A}(t)\Delta t \tag{5.51}$$

This equation can be applied iteratively to obtain $s_A(t + \Delta t)$ at a next time from its value $s_A(t)$ at a previous time. For example the next step is:

$$s_A(t + 2\Delta t) = s_A(t + \Delta t) + v_{s,A}(t + \Delta t)\Delta t \tag{5.52}$$

This integration procedure, which is called the *Euler forward method*, can be iterated to get the complete numerical integral for s_A for all times. Reducing the time step Δt helps to increase the accuracy. The method can also be applied to integrate acceleration and velocity vectors, by performing scalar integration along each of the axes in a Cartesian system like in eq. (5.57). There are multiple methods for numerically solving integrals and obtaining derivatives. An important integration method is Euler's forward method, which is the simplest but by far not the most efficient and accurate method. To be sure of a result, it can be a good strategy to solve a problem both analytically and numerically. Then the numerical solution can function as a plausibility check for the analytical solution or vice versa.

5.9 Kinematic analysis and coordinate systems

The kinematic analysis presented in the previous sections is valid for any coordinate system or constraint curve $\vec{\boldsymbol{r}}_s$. We now show how to obtain the 3 dimensional motion in a specific coordinate system or a specific path curve $\vec{\boldsymbol{r}}_s(s)$ that is expressed in Cartesian or cylindrical coordinates. An important case is motion along a straight line.

5.9.1 Rectilinear motion

Motion along a straight 1D line is the most simple to analyse. If motion occurs along the x-coordinate axis in Cartesian coordinates, the path curve pointing in the positive x-coordinate direction is:

$$\vec{\boldsymbol{r}}_s(s) = s\hat{\boldsymbol{\imath}} \tag{5.53}$$

A certain point mass i with path coordinate s_i is located at position vector $\vec{\boldsymbol{r}}_i = \vec{\boldsymbol{r}}_s(s_i) = s_i\hat{\boldsymbol{\imath}}$. Projecting its position on the x-coordinate axis, we find that its x-coordinate is identical to its path coordinate $x_i = \vec{\boldsymbol{r}}_i \cdot \hat{\boldsymbol{\imath}} = s_i$. So, essentially everything that we have derived for $s_i(t)$ also holds for the x-coordinate $x_i(t)$ of a point mass that moves on a rectilinear path along the x-coordinate axis, with $s_i = x_i$, $v_{s,i} = \dot{x}_i$ and $a_{s,i} = \ddot{x}_i$, from Eqs. (5.30)–(5.32).

5.9.2 Kinematics in Cartesian coordinates

Cartesian coordinates have the advantage that the direction of the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ is independent of the coordinates in the system. If the coordinate system is not rotating (see also Sec. 6.12), the direction of the unit vectors is also independent of time $\frac{\mathrm{d}\hat{\boldsymbol{\imath}}}{\mathrm{d}t} = \frac{\mathrm{d}\hat{\boldsymbol{\jmath}}}{\mathrm{d}t} = \frac{\mathrm{d}\hat{\boldsymbol{k}}}{\mathrm{d}t} = \vec{\boldsymbol{0}}$. This greatly simplifies the kinematic analysis and allows describing the motion of a point mass i using its 3 time-dependent coordinates x_i, y_i and z_i and Eqs. (5.28)–(5.29) as follows:

$$\vec{\boldsymbol{r}}_i(t) = x_i(t)\ \hat{\boldsymbol{\imath}} + y_i(t)\ \hat{\boldsymbol{\jmath}} + z_i(t)\ \hat{\boldsymbol{k}} \tag{5.54}$$
$$\vec{\boldsymbol{v}}_i(t) = \dot{x}_i(t)\ \hat{\boldsymbol{\imath}} + \dot{y}_i(t)\ \hat{\boldsymbol{\jmath}} + \dot{z}_i(t)\ \hat{\boldsymbol{k}} \tag{5.55}$$
$$\vec{\boldsymbol{a}}_i(t) = \ddot{x}_i(t)\ \hat{\boldsymbol{\imath}} + \ddot{y}_i(t)\ \hat{\boldsymbol{\jmath}} + \ddot{z}_i(t)\ \hat{\boldsymbol{k}} \tag{5.56}$$

It is also possible to integrate the individual functions $\ddot{x}_i(t)$, $\ddot{y}_i(t)$, $\ddot{z}_i(t)$ to obtain the velocity and position of the point mass when the acceleration is known, similar to the integrals shown in Eqs. (5.35), (5.36), for the velocity and acceleration component along the x, y and z-axis:

$$\vec{\boldsymbol{r}}_i(t_2) = \vec{\boldsymbol{r}}_i(t_1) + \int_{t_1}^{t_2} \dot{x}_i(t)\mathrm{d}t\ \hat{\boldsymbol{\imath}} + \int_{t_1}^{t_2} \dot{y}_i(t)\mathrm{d}t\ \hat{\boldsymbol{\jmath}} + \int_{t_1}^{t_2} \dot{z}_i(t)\mathrm{d}t\ \hat{\boldsymbol{k}} \tag{5.57}$$
$$\vec{\boldsymbol{v}}_i(t_2) = \vec{\boldsymbol{v}}_i(t_1) + \int_{t_1}^{t_2} \ddot{x}_i(t)\mathrm{d}t\ \hat{\boldsymbol{\imath}} + \int_{t_1}^{t_2} \ddot{y}_i(t)\mathrm{d}t\ \hat{\boldsymbol{\jmath}} + \int_{t_1}^{t_2} \ddot{z}_i(t)\mathrm{d}t\ \hat{\boldsymbol{k}} \tag{5.58}$$

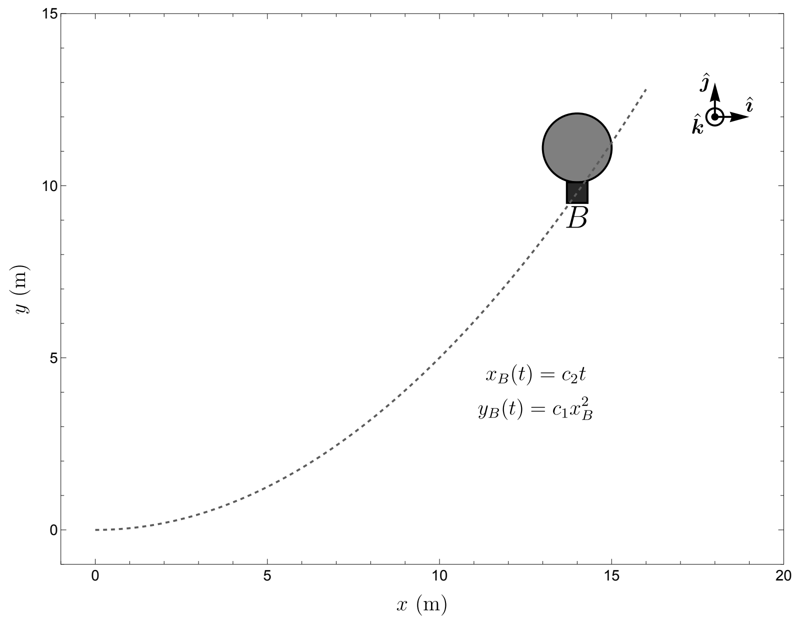


Figure 5.9: Example 5.3, determining the velocity and acceleration of a balloon along a parabolic path curve with $c_2 = 1\,\mathrm{m/s}$ and $c_1 = \frac{1}{20}\mathrm{m}^{-1}$.

S **Example 5.3** Let us illustrate the kinematic analysis method in Cartesian coordinates by determining the acceleration and velocity vector of the balloon B in Fig. 5.9, that moves along a parabolic path curve with $y_B = c_1 x_B^2$ in the xy-plane. The motion of the balloon is described by:

$$\begin{aligned} x_B(t) &= c_2 t \\ y_B(t) &= c_1 c_2^2 t^2 \end{aligned}$$

The position vector of the balloon can be differentiated twice, like in Eq. (5.54) and Eq. (5.56), to obtain the velocity and acceleration vectors as follows:

$$\begin{aligned} \vec{\boldsymbol{r}}_B(t) &= c_2 t\ \hat{\boldsymbol{\imath}} + c_1 t^2\ \hat{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{v}}_B(t) &= c_2\ \hat{\boldsymbol{\imath}} + 2c_1 t\ \hat{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{a}}_B(t) &= 2c_1\ \hat{\boldsymbol{\jmath}} \end{aligned}$$

The speed can be obtained by taking the absolute value of the velocity vector:

$$v_B = |\vec{\boldsymbol{v}}_B| = \sqrt{\vec{\boldsymbol{v}}_B \cdot \vec{\boldsymbol{v}}_B} = \sqrt{c_2^2 + (2c_1 t)^2} \tag{5.59}$$

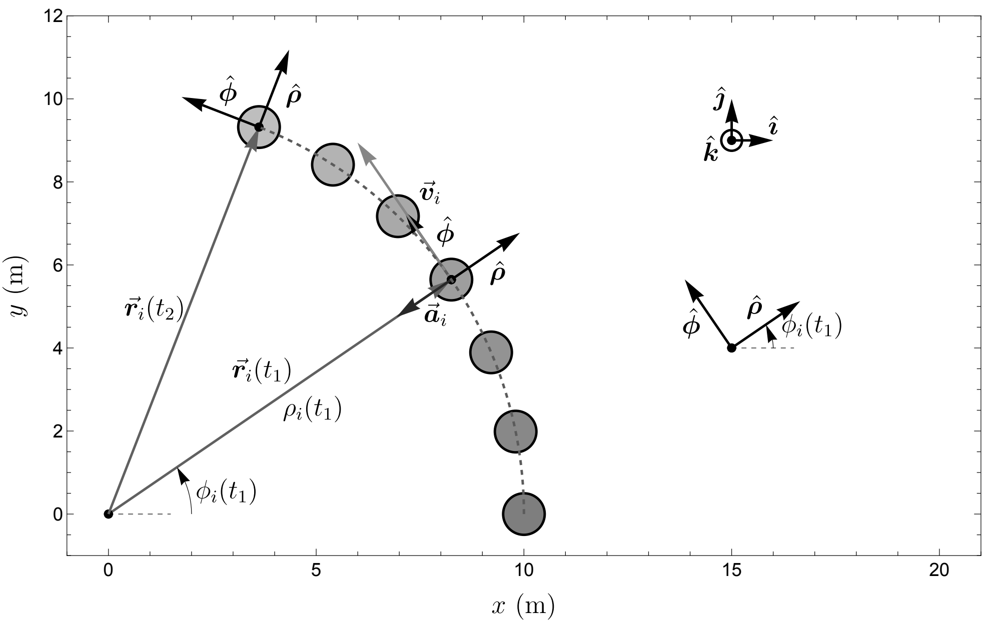


Figure 5.10: Point mass i moving in a circular orbit in the xy-plane. Its position is indicated by cylindrical coordinates ρ_i and ϕ_i. Its velocity vector (orange) points in the $\hat{\boldsymbol{\phi}}$ direction, tangential to its path curve, its acceleration vector (purple) is in the negative $\hat{\boldsymbol{\rho}}$ direction.

5.9.3 Kinematics in cylindrical coordinates

Cylindrical coordinates, which were introduced in section 3.3.2 are often convenient when the kinematics involves rotation around an axis, like in Fig. 5.10. The position, velocity and acceleration vector in cylindrical coordinates can be written as:

$$\vec{\boldsymbol{r}}_i = \rho_i\ \hat{\boldsymbol{\rho}} + z_i\ \hat{\boldsymbol{k}} \tag{5.60}$$

$$\vec{\boldsymbol{v}}_i = v_{i,\rho}\ \hat{\boldsymbol{\rho}} + v_{i,\phi}\ \hat{\boldsymbol{\phi}} + v_{i,z}\ \hat{\boldsymbol{k}} \tag{5.61}$$

$$\vec{\boldsymbol{a}}_i = a_{i,\rho}\ \hat{\boldsymbol{\rho}} + a_{i,\phi}\ \hat{\boldsymbol{\phi}} + a_{i,z}\ \hat{\boldsymbol{k}} \tag{5.62}$$

Concept. *Kinematics in cylindrical coordinates*
We will show that by taking the time derivatives of the position vector, the components of the velocity vector and acceleration vector in Eqs. (5.60–5.62) are found to be:

$$v_{i,\rho} = \dot{\rho}_i \tag{5.63}$$

$$v_{i,\phi} = \rho_i\dot{\phi}_i \tag{5.64}$$

$$v_{i,z} = \dot{z}_i \tag{5.65}$$

$$a_{i,\rho} = \ddot{\rho}_i - \rho_i\dot{\phi}_i^2 \tag{5.66}$$

$$a_{i,\phi} = 2\dot{\rho}_i\dot{\phi}_i + \rho_i\ddot{\phi}_i \tag{5.67}$$

$$a_{i,z} = \ddot{z}_i \tag{5.68}$$

Derivation.

The determination of these velocity and acceleration vectors in cylindrical coordinates is more difficult because the direction of the unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ is not constant. To determine their time derivatives, we use that the position vector can also be expressed in Cartesian unit vectors as:

$$\vec{\boldsymbol{r}}_i = \rho_i \cos\phi_i\hat{\boldsymbol{\imath}} + \rho_i \sin\phi_i\hat{\boldsymbol{\jmath}} + z_i\hat{\boldsymbol{k}} \tag{5.69}$$

From the directions of the unit vectors, as shown in Fig. 5.10, the cylindrical unit vectors can be expressed in Cartesian unit vectors as:

$$\hat{\boldsymbol{\rho}} = \cos\phi_i\hat{\boldsymbol{\imath}} + \sin\phi_i\hat{\boldsymbol{\jmath}} \tag{5.70}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi_i\hat{\boldsymbol{\imath}} + \cos\phi_i\hat{\boldsymbol{\jmath}} \tag{5.71}$$

Now it can be seen that the time derivatives of the unit vectors obey (chain rule):

$$\frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}t} = (-\sin\phi_i\hat{\boldsymbol{\imath}} + \cos\phi_i\hat{\boldsymbol{\jmath}})\dot{\phi}_i = \dot{\phi}_i\ \hat{\boldsymbol{\phi}} = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\rho}} \tag{5.72}$$

$$\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} = (-\cos\phi_i\hat{\boldsymbol{\imath}} - \sin\phi_i\hat{\boldsymbol{\jmath}})\dot{\phi}_i = -\dot{\phi}_i\ \hat{\boldsymbol{\rho}} = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\phi}}, \tag{5.73}$$

where the terms in brackets were replaced by $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\rho}}$ using Eqs. (5.70) and (5.71), and the equations were further simplified using the vector $\vec{\boldsymbol{\omega}} = \dot{\phi}_i\hat{\boldsymbol{k}}$. We use these time derivatives of the unit vectors to find the velocity and acceleration in cylindrical coordinates, and the components given in Eqs. (5.63–5.68) by differentiating twice:

$$\vec{\boldsymbol{r}}_i = \rho_i\ \hat{\boldsymbol{\rho}} + z_i\ \hat{\boldsymbol{k}} \tag{5.74}$$

$$\vec{\boldsymbol{v}}_i = \frac{\mathrm{d}\vec{\boldsymbol{r}}_i}{\mathrm{d}t} = \dot{\rho}_i\ \hat{\boldsymbol{\rho}} + \rho_i\frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}t} + \dot{z}_i\ \hat{\boldsymbol{k}} \tag{5.75}$$

$$= \dot{\rho}_i\ \hat{\boldsymbol{\rho}} + \rho_i\dot{\phi}_i\ \hat{\boldsymbol{\phi}} + \dot{z}_i\ \hat{\boldsymbol{k}} \tag{5.76}$$

$$\vec{\boldsymbol{a}}_i = \frac{\mathrm{d}\vec{\boldsymbol{v}}_i}{\mathrm{d}t} = \ddot{\rho}_i\hat{\boldsymbol{\rho}} + \dot{\rho}_i\frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}t} + \dot{\rho}_i\dot{\phi}_i\hat{\boldsymbol{\phi}} + \rho_i\ddot{\phi}_i\hat{\boldsymbol{\phi}} + \rho_i\dot{\phi}_i\frac{\mathrm{d}\hat{\boldsymbol{\phi}}}{\mathrm{d}t} + \ddot{z}_i\hat{\boldsymbol{k}} \tag{5.77}$$

$$= (\ddot{\rho}_i - \rho_i\dot{\phi}_i^2)\hat{\boldsymbol{\rho}} + (2\dot{\rho}_i\dot{\phi}_i + \rho_i\ddot{\phi}_i)\hat{\boldsymbol{\phi}} + \ddot{z}_i\hat{\boldsymbol{k}} \tag{5.78}$$

The components of these vectors are identical to the equations (5.63–5.68) thus proving the correctness of those equations.

5.9.4 Circular motion

A special important case of constrained motion on a path curve is motion on a circle with constant radius ρ. So, in Fig. 5.10 this means ρ_i = constant. The components of the velocity and acceleration vector can be found by substituting $\dot{\rho}_i = 0$ and $\ddot{\rho}_i = 0$ into Eqs. (5.75–5.78) to obtain:

$$v_{i,\rho} = 0 \tag{5.79}$$

$$v_{i,\phi} = \rho\dot{\phi}_i = \dot{s}_i \tag{5.80}$$

$$a_{i,\rho} = -\rho\dot{\phi}_i^2 = -\frac{v_{s,i}^2}{\rho} \tag{5.81}$$

$$a_{i,\phi} = \rho\ddot{\phi}_i = \dot{v}_{s,i} = \ddot{s}_i \tag{5.82}$$

It can be seen that the velocity vector only has a component in the $\hat{\boldsymbol{\phi}}$ direction, tangential to the circle. The acceleration has both a tangential component $a_{i,\phi}$, and a radial or centripetal component $a_{i,\rho}$ that points towards the centre of the circle. The time derivatives of the angle $\omega_{o,i} = \dot{\phi}_i$ and $\alpha_{o,i} = \ddot{\phi}_i$ are called the *orbital angular velocity* and *orbital angular acceleration* of the point mass. The word orbital is used to distinguish this angular velocity from the angular velocity of a rigid body, which is sometimes called the *spin angular velocity* and will be discussed later in Ch. 9.

5.9.5 Natural t,n,b coordinates for curvilinear motion

Natural coordinates make use of the fact that even if a curved trajectory is not really a circle, it is possible to *approximate* it by a circle with radius ρ at every point of the trajectory, as shown in Fig. 5.11. Like in the previous subsection, a cylindrical coordinate system can be chosen such that the centre of that circle (the *centre of curvature* O_i) is at the origin and the point mass position is determined by a path coordinate $s_i = \rho\phi_i$, where ρ is called the *radius of curvature*. The plane in which the instantaneous circle is positioned is called the *osculating plane*. For an instantaneous moment, the motion of the point mass can be described using *natural coordinates* or t, n, b coordinates, that are defined by a unit vector $\hat{\boldsymbol{t}}$ that is tangential to the curvilinear path[2], a normal unit vector $\hat{\boldsymbol{n}}$ that points towards the centre of the instantaneous circle and a binormal unit vector $\hat{\boldsymbol{b}} = \hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}$, that points perpendicular to the osculating plane. At every instant of time the path curve can have a different origin O and radius ρ. Note that in Fig. 5.11 the origin changes from point O_1 to O_2. As a consequence, at times t_1 and times t_3, the directions of both the $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{b}}$ unit vectors have flipped because the motion changed from anti-clockwise to clockwise, even though the tangential unit vector $\hat{\boldsymbol{t}}$ is the same. In t,n,b coordinates, the components of the instantaneous velocity vector are $\vec{\boldsymbol{v}}_i = v_{i,t}\hat{\boldsymbol{t}} + v_{i,n}\hat{\boldsymbol{n}}$ and the acceleration vector is $\vec{\boldsymbol{a}}_i = a_{i,t}\hat{\boldsymbol{t}} + a_{i,n}\hat{\boldsymbol{n}}$. Note the sign change $a_{i,n} = -a_{i,\rho}$ compared to Eq. (5.81), because $\hat{\boldsymbol{n}}$ points inward, while $\hat{\boldsymbol{\rho}}$ points away from the rotation axis. The velocity and acceleration components are:

$$v_{i,t} = \dot{s}_i \tag{5.83}$$

$$v_{i,n} = 0 \tag{5.84}$$

$$a_{i,t} = \dot{v}_{s,i} = \ddot{s}_i \tag{5.85}$$

$$a_{i,n} = v_{s,i}^2/\rho \tag{5.86}$$

[2]The unit vector $\hat{\boldsymbol{t}}$ is identical to the unit vector $\hat{\boldsymbol{s}}$ in path coordinates. If the curvilinear motion is a counterclockwise circle around the z-axis one also has $\hat{\boldsymbol{b}} = \hat{\boldsymbol{k}}$, $\hat{\boldsymbol{t}} = \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{\rho}}$ in cylindrical coordinates.

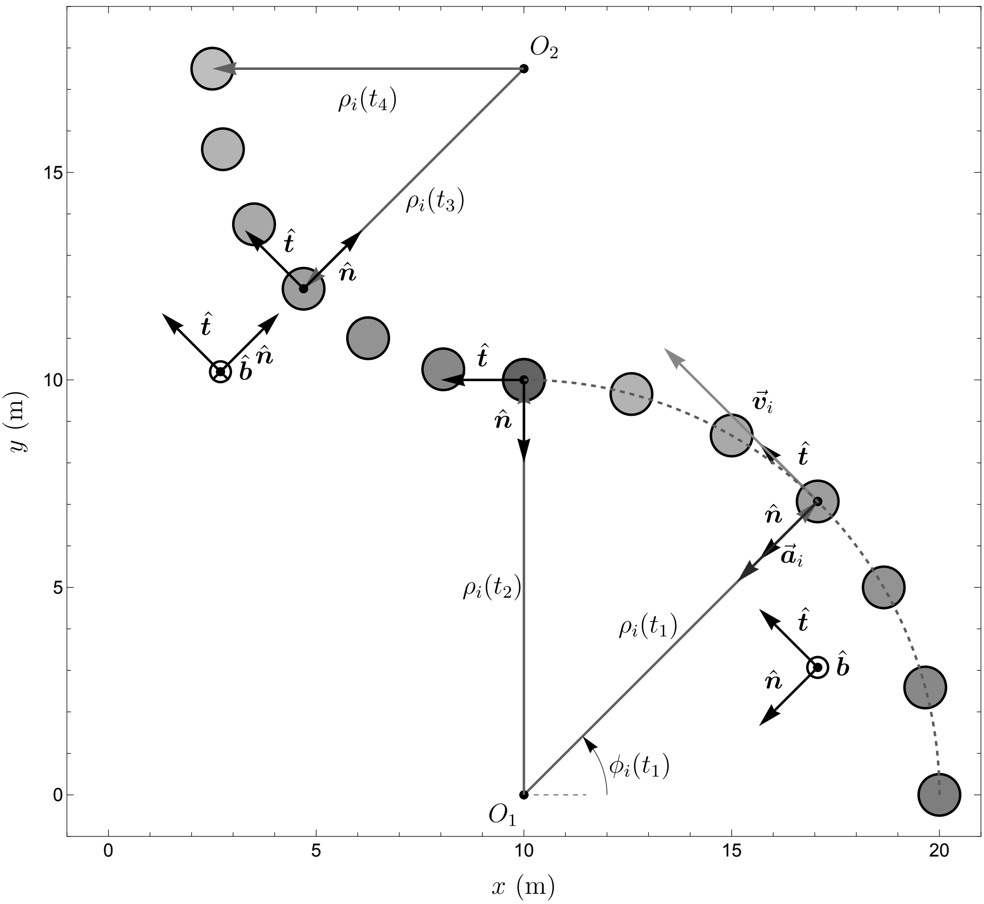


Figure 5.11: Natural t,n,b coordinates that define the local unit vectors depending on the shape of the path curve.

Velocity and acceleration along the $\hat{\boldsymbol{b}}$ direction are always zero. Note that these equations are very similar to those for circular motion, except for the minus sign in Eq. (5.81). The radius ρ can change continuously along the curvilinear path and needs to be determined at every point of the curve in order to use these equations to determine the given velocity and acceleration components. Furthermore, because they do not have a fixed origin, natural coordinates are not used for describing positions, and are therefore not a replacement for Cartesian and cylindrical coordinates.

5.9.6 Summary parametrised motion and kinematics

After having discussed kinematics in different coordinate systems we conclude by summarising the different ways these coordinates allow us to parametrise motion in 3D and obtain the kinematics. Depending on the situation, the time dependent motion $\vec{\boldsymbol{r}}(t)$ can be fully described either using 3 time dependent coordinate functions $x(t), y(t), z(t)$, or using the path curve $\vec{\boldsymbol{r}}_s(s) = x(s)\hat{\boldsymbol{\imath}} + y(s)\hat{\boldsymbol{\jmath}} + z(s)\hat{\boldsymbol{k}}$ and a single time dependent coordinate function $s(t)$:

Parametrisations of motion and determination of $\vec{v}$ and $\vec{a}$

- **The time dependent coordinate functions**

1. Cartesian coordinate functions $x(t)$ and $y(t)$.
   –Use methods from Sec. 5.9.2 to obtain $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$.
2. Cylindrical coordinate functions $\rho(t)$ and $\phi(t)$.
   –Use methods from Sec. 5.9.3 to obtain $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$.
   –First convert to Cartesian coordinates using $x(t) = \rho(t)\cos\phi(t)$ and $y(t) = \rho(t)\sin\phi(t)$, and proceed like in 1.

- **The path curve and one coordinate function of time**

3. Cartesian path curve $x(s), y(s)$, and path coordinate $s(t)$.
   –Use methods from Sec. 5.8 to determine the path velocity v_s, and tangential component of acceleration a_s from $s(t)$.
   –Use t, n, b coordinates (Sec. 5.9.5) to obtain the normal component of acceleration a_n by first determining the local path radius ρ.
   –Alternatively one can determine $x(t) = x(s(t))$, $y(t) = y(s(t))$ and proceed like in 1.
4. Cartesian path curve $y(x)$ and coordinate function $x(t)$.
   –Determine the function $y(t) = y(x(t))$ and use method 1.
5. Cylindrical path curve $\rho(s), \phi(s)$ and path coordinate $s(t)$.
   –Proceed using $s(t)$ and t, n, b coordinates like in 3.
   –Use $x(t) = \rho(s(t))\cos\phi(s(t))$ and $y(t) = \rho(s(t))\sin\phi(s(t))$, and proceed

like in 1.

6. Cylindrical path curve $\rho(\phi)$ and coordinate function $\phi(t)$.
   –Determine the function $\rho(t) = \rho(\phi(t))$ and use method 2.

As becomes clear from this list, there are often multiple methods to determine the velocity and acceleration from given kinematic information. However, some methods might be much more mathematically complex than others. It is therefore useful to become familiar with the different methods, and get experience in choosing the easiest one in a certain situation.

5.9.7 Determining the path coordinate*

We note that it is also possible to determine the path coordinate $s(t)$ by going from parametrisation 4 to 3. For that purpose you use that it holds from Pythagoras' theorem that the distance $\mathrm{d}s$ along a straight infinitesimal line element obeys $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, such that $s(t)$ can be determined using the integral:

$$s(t) = \int_0^{s(t)} \mathrm{d}s = \int_0^{x(t)} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2} \mathrm{d}x \tag{5.87}$$

For example if we deal with motion at constant speed along a straight tilted line, $y = \alpha x$, and $x(t) = v_x t$ then we have:

$$s(t) = \int_0^{v_x t} \sqrt{1 + \alpha^2} \mathrm{d}x = \sqrt{1 + \alpha^2} \times v_x t \tag{5.88}$$

Noting that the y velocity component is $v_y = \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}x}\dot{x} = \alpha v_x$, we see that the path velocity $v_s = \dot{s} = \sqrt{1 + \alpha^2} \times v_x = \sqrt{v_x^2 + v_y^2}$ as expected.

5.9.8 Determining ρ, the radius of curvature*

When working with natural t, n, b coordinates, one needs to know the local radius of curvature ρ at each position to determine the velocity and acceleration components using (5.83-5.86). However, determining the radius of curvature ρ is not always easy. For a path curve in the xy-plane that can be written in the form $y(x)$, the instantaneous radius ρ can be determined using this convenient function:

$$\rho = \frac{\left[1 + \frac{\mathrm{d}y}{\mathrm{d}x}^2\right]^{\frac{3}{2}}}{\left|\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|} \tag{5.89}$$

Derivation. *This equation can be derived by considering a circle with centre* (x_0, y_0) *and radius* ρ *while ensuring that its first and second derivative with respect to* x *match those of the path curve* $y(x)$ *at the point* (x, y) *where they touch:*

$$\rho^2 = (y - y_0)^2 + (x - x_0)^2 \tag{5.90}$$

$$y = y_0 + \sqrt{\rho^2 - (x - x_0)^2} \tag{5.91}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x - x_0}{y - y_0} \tag{5.92}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-1}{y - y_0} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{x - x_0}{(y - y_0)^2} = -\frac{1 + \frac{\mathrm{d}y}{\mathrm{d}x}^2}{y - y_0} \tag{5.93}$$

$$y - y_0 = -\frac{1 + \frac{\mathrm{d}y}{\mathrm{d}x}^2}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}} \tag{5.94}$$

$$\rho^2 = (y - y_0)^2\left[1 + \frac{(x - x_0)^2}{(y - y_0)^2}\right] \text{(from Eq. (5.90))} \tag{5.95}$$

$$= \left[\frac{1 + \frac{\mathrm{d}y}{\mathrm{d}x}^2}{|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}|}\right]^2 \left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right] \text{(used (5.92)\&(5.94))} \tag{5.96}$$

$$\rho = \frac{(1 + \frac{\mathrm{d}y}{\mathrm{d}x}^2)^{3/2}}{|\frac{\mathrm{d}^2y}{\mathrm{d}x^2}|} \tag{5.97}$$

5.10 Segmented kinematics

Sometimes motion occurs in different time segments or space segments. For example a car might accelerate, and at a certain time t_1 it brakes and continues at a constant velocity (see Fig. 5.12). This kind of motion is called segmented motion and can be described by segmented or piece-wise functions like this:

$$s_A(t) = \begin{cases} f_1(t) & t \le t_1 \\ f_2(t) & t > t_1 \end{cases} \tag{5.98}$$

The velocity and acceleration can be found by differentiation on the different segments as follows:

$$v_{s,A}(t) = \dot{s}_A(t) = \begin{cases} \dot{f}_1(t) & t \le t_1 \\ \dot{f}_2(t) & t > t_1 \end{cases} \tag{5.99}$$

$$a_{s,A}(t) = \ddot{s}_A(t) = \begin{cases} \ddot{f}_1(t) & t \le t_1 \\ \ddot{f}_2(t) & t > t_1 \end{cases} \tag{5.100}$$

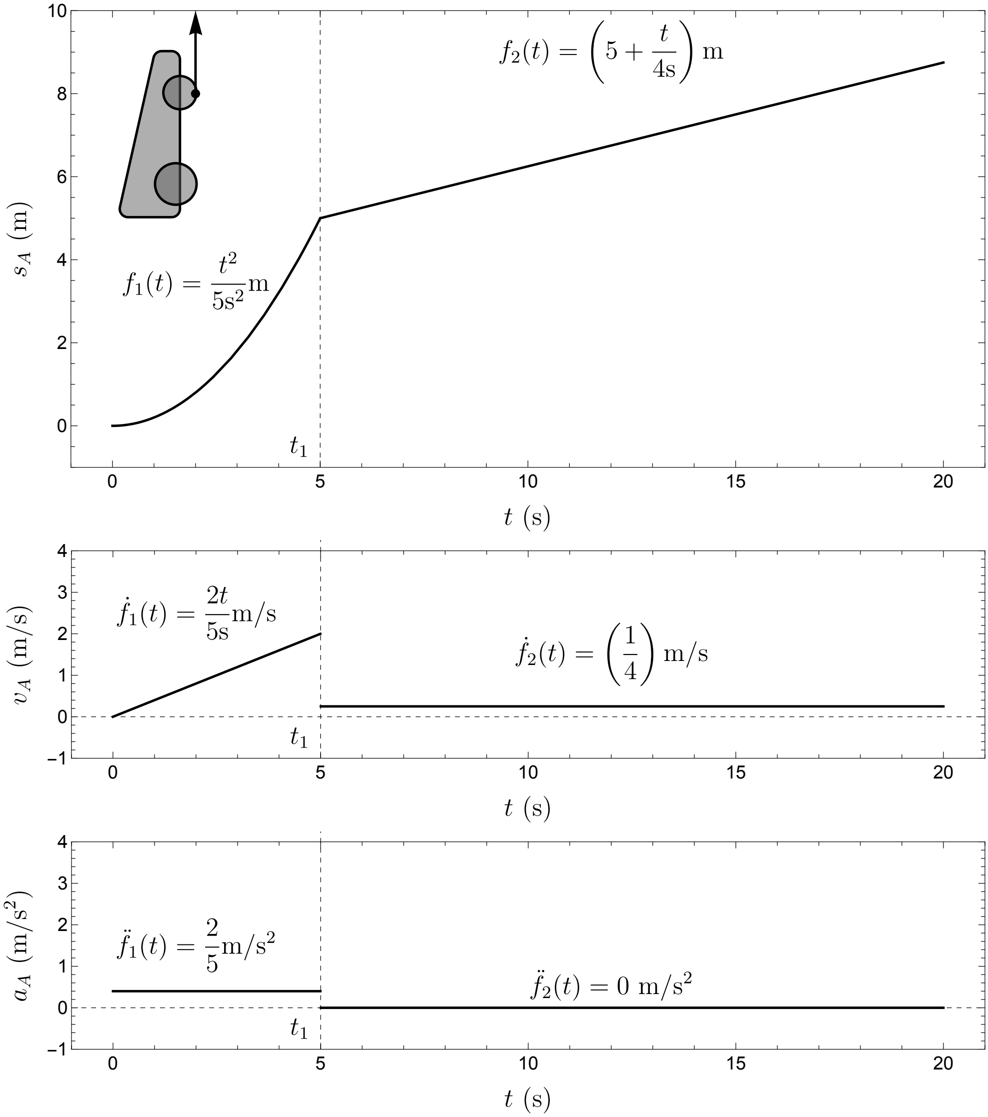


Figure 5.12: Segmented kinematics, after $t = t_1$ the motion is described by $f_2(t)$ instead of $f_1(t)$.

The procedure is similar if the motion is segmented along the path coordinate s instead of time t, using functions $f_1(s)$ and $f_2(s)$.

When integrating the segmented motion, one proceeds by integrating of the first segment, and then taking the end position and velocity of the first segment as initial condition for the integration of the next segment. It is important to note that it is impossible for a point mass to instantaneously 'jump' or 'teleport' in space. For that reason $f_1(t_1) = f_2(t_1)$ needs to be obeyed. Moreover, from Newton's second law and the inertia of mass, we know that it also takes time to change the velocity of an object. Therefore often also $\dot{f}_1(t_1) = \dot{f}_2(t_1)$ holds for the velocities to be equal at the interface between the segments. There are exceptions in cases like collisions, where very high forces occur, such that the accelerations are so high that a substantial velocity change can occur suddenly.

5.11 Summary

Let us summarise the kinematic methods that have been discussed in this chapter.

- **Constraint equations**
  - Using geometry for determining constraint equations.
  - Taking time derivatives of constraint equations for the position, velocity and acceleration of point masses and objects. See e.g. Eq. (5.5).
- **Equations of motion**
  - Solving equations of motion using initial conditions and constraint equations.
- **Kinematics along the path curve**
  - Integration and differentiation with respect to time to obtain the path coordinate $s(t)$, path velocity $v_s(t)$ and path acceleration $a_s(t)$.
  - Integration and differentiation with respect to path coordinate s to obtain the functions $v_s(s)$ and $a_s(s)$.
  - Analysing segmented kinematics along the path curve using (5.98), (5.99) and (5.100).
  - Integration and differentiation with respect to the path velocity $a_s(v)$.
- **Kinematics in 3D coordinate systems**

- Describing a path curve and the motion with the position vector $\vec{\boldsymbol{r}}(t)$ of a point mass in 3D using Cartesian and cylindrical coordinates and work with natural t, n, b coordinates in different parametrisations.
- Properly draw coordinate systems, path coordinates and unit vectors. Draw position, velocity and acceleration vectors and project them on the coordinate axes, obtaining their components, magnitudes and angles.
- Determine the velocity vector $\vec{\boldsymbol{v}}(t)$ and acceleration vector $\vec{\boldsymbol{a}}(t)$ for a given motion $\vec{\boldsymbol{r}}(t)$, irrespective of its parametrisation (see Sec. 5.9.6). In Cartesian coordinates use (5.54-5.56), in cylindrical coordinates use (5.60-5.68), and in natural (t, n, b) coordinates use (5.83-5.86).
- Determine all components of the position and velocity vectors by integrating twice for a given $\vec{\boldsymbol{a}}(t)$ in Cartesian coordinates by using Eq. (5.57) and Eq. (5.58).
- Know that the direction of the velocity vector $\vec{\boldsymbol{v}}$ is always tangential to the path curve, while the acceleration vector $\vec{\boldsymbol{a}}$ is not.
- Solve kinematic problems in 3D using numerical integration and differentiation.

In the next chapter we will discuss the kinetics of point masses, deriving their dynamics from the effect of forces via Newton's second law.

6 Kinetics of Point Masses

In the previous chapter we have discussed and developed a toolbox to analyse the *kinematics* of point masses, determining their motion, velocity and acceleration in various coordinate systems. In this chapter we extend our toolbox by determining the effect of forces on acceleration, thus analysing the *kinetics* of point masses.

The outline of this chapter is as follows. We first introduce Newton's second law (Sec. 6.1) and use it to define mass (Sec. 6.2). We define force vectors and their properties governed by Newton's third law (Sec. 6.3). Then we describe in detail the step-by-step procedure (Sec. 6.4) to determine and solve the equations of motion. The procedure starts from a sketch (Sec. 6.5) and choosing the coordinate system (Sec. 6.6). Then constraint equations are determined, free body diagrams are used to determine the resultant forces, and Newton's second law is used to determine the equations of motion along each coordinate axis (Sec. 6.7-6.10), which are then solved (Sec. 6.11.1) to obtain the motion or forces (Sec. 6.11.2). Inertial reference frames are introduced (Sec. 6.12) and finally we discuss the properties of various forces (Sec. 6.13).

6.1 Newton's second law

Almost all motion and dynamics you see around you can be derived from Newton's second law and the characteristics of the four fundamental forces of nature. It is amazing that so many phenomena in our daily life can be described by such a deceptively simple equation. There are several reasons for this. First of all, Newton's laws are not as simple as they seem, and require careful use of vectors, mathematics and kinematic techniques. Secondly, the force term can have many different appearances, leading to a wide variety of motion, and thirdly, most importantly, dynamics can become very complex and difficult to predict when many point masses are present.

Let us have a closer look at Newton's second law, Eq. (4.2):

$$\sum_j \vec{\boldsymbol{F}}_{ij} = m_i \vec{\boldsymbol{a}}_i \tag{6.1}$$

We know from the previous chapter how to interpret the acceleration vector $\vec{\boldsymbol{a}}_i$, however what is the precise meaning of the scalar mass m_i and the force vectors $\vec{\boldsymbol{F}}_{ij}$?

6.2 Defining mass

Definition. *Mass*
What is the mass of an object? Newton's second law tells us that if there is a constant force $\vec{\boldsymbol{F}}_{\text{ref}}$ acting on the object, and we measure an acceleration vector $\vec{\boldsymbol{a}}_i$, the object's mass is given by:

$$m_i \equiv \frac{|\vec{\boldsymbol{F}}_{\text{ref}}|}{|\vec{\boldsymbol{a}}_i|}, \tag{6.2}$$

where we used the $\equiv$ sign to indicate that this is the definition of the mass, or *inertial* mass, of the object. The object's mass obeys Eq. (6.2) irrespective of the properties of the force $\vec{\boldsymbol{F}}_{\text{ref}}$, and is thus an intrinsic property of the object that normally does not change. To assign a value to the mass we use the international systems of units, the SI units. The previous equation tells us that if we apply a force of 1 N and observe an acceleration of 1 m/s^2, the mass of the object is 1 kg.

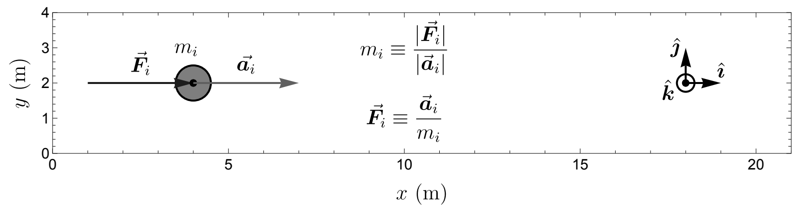


Figure 6.1: Newton's second law defines both the force vector $\vec{\boldsymbol{F}}_i$ acting on a point mass and its mass m_i.

6.2.1 Weight of a mass

Since measuring accelerations is not always easy, in practice we often determine the mass of an object via the gravitational force acting on the object, e.g. by using a weighing scale. This is possible because it was experimentally determined that the gravitational force on earth is proportional to the mass of the object: $\vec{\boldsymbol{F}}_{i,g} = m_i\vec{\boldsymbol{g}}$. The magnitude of the gravitational force on the object on earth's surface is defined as the weight W of the object, so we can write:

$$W_i \equiv m_i |\vec{\boldsymbol{g}}| \tag{6.3}$$

Here, the gravitational acceleration vector $\vec{\boldsymbol{g}}$ has a magnitude of approximately $9.8\,\mathrm{m/s^2}$ on earth and points towards the center of the earth. The *gravitational* mass of an object can be determined using the equation $m_i = W_i/|\vec{\boldsymbol{g}}|$ and has experimentally been found to be equal to its *inertial* mass as determined from Newton's second law.

6.2.2 Point mass

Concept. *Point mass*
A point mass i is an object that behaves like a mathematical point in space at position vector $\vec{\boldsymbol{r}}_i$ and has mass m_i.

Since it is located at a single point, a point mass can only have a single position, velocity and acceleration vector. The point mass is an important concept, because Newton's laws strictly only hold for point masses. Moreover, the smallest known particles, the elementary particles, behave very similar to point masses, and all larger objects are made out of elementary particles. Also, as we will see, the dynamics of larger objects can often still be well approximated by treating them as a point mass, therefore the term point mass is also used for larger objects that move like a point mass, e.g. because they do not rotate.

6.3 What is force?

6.3.1 Force and Newton's second law

Definition. *Force*
Just like mass, the concept of force can be defined[1] via Newton's second law:

$$\sum_j \vec{\boldsymbol{F}}_{ij} \equiv m_i \vec{\boldsymbol{a}}_i \tag{6.4}$$

So if an object with mass m_i is observed to accelerate with an acceleration vector $\vec{\boldsymbol{a}}_i$, Newton's second law tells us that the sum of the force vectors acting

[1] One might ask how it is possible to define both mass and force from Newton's second law. The answer is that mass is a property of the object that accelerates, whereas forces are due to interactions between objects. By properly designed experiments the characteristics of the force and mass can be determined independently in most cases.

on the object is $\sum_i \vec{\boldsymbol{F}}_{ij}$. To determine individual force vectors $\vec{\boldsymbol{F}}_{ij}$ (instead of their sum) one should perform experiments where only one force is acting on m_i.

6.3.2 Force and Newton's third law

Newton's third law gives us information on the way forces act, since it holds for all known forces. It states that all (fundamental) forces act between two point masses. If there are two point masses m_i and m_j, the force vector $\vec{\boldsymbol{F}}_{ij}$ generated on mass m_i by mass m_j is always equal in magnitude and opposite in direction to another force vector $\vec{\boldsymbol{F}}_{ji}$ that is generated on m_j by mass m_i. In vector notation, as shown in Fig. 6.2:

$$\vec{\boldsymbol{F}}_{ij} = -\vec{\boldsymbol{F}}_{ji} \tag{6.5}$$

Moreover, these two force vectors are collinear (i.e. have the same line of action) to the relative position vector $\vec{\boldsymbol{r}}_{j/i} = \vec{\boldsymbol{r}}_j - \vec{\boldsymbol{r}}_i$ connecting the two point masses. Newton's third law, Eq. (6.5), also shows us that a force $\vec{\boldsymbol{F}}_{ij}$ can never exist alone, and is always accompanied by a reaction force $\vec{\boldsymbol{F}}_{ji}$ acting on another point mass, which sometimes can be left out of the analysis for simplicity. In words Newton's third law is often stated as 'action force is equal and opposite to reaction force'.

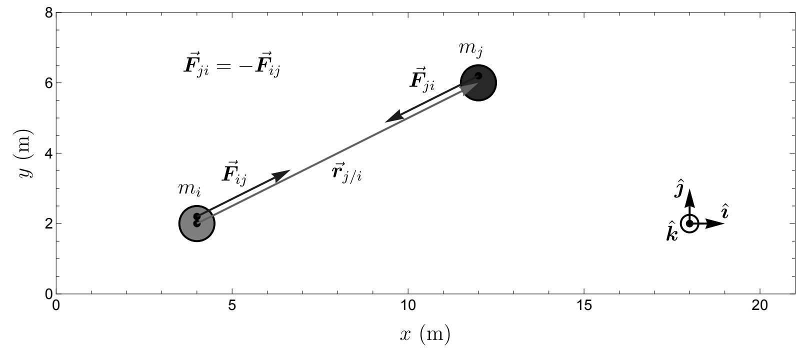


Figure 6.2: The third law of Newton states that for every force vector $\vec{\boldsymbol{F}}_{ij}$ that acts on a point mass i, there is another force $\vec{\boldsymbol{F}}_{ji}$ that acts on another point mass. These two force vectors have equal magnitude, are collinear and have opposite direction ($\vec{\boldsymbol{F}}_{ji} = -\vec{\boldsymbol{F}}_{ij}$)

.

6.3.3 Properties of forces

Newton's third law tells us something about the properties of forces between point masses. There are only four known fundamental forces, which are the gravitational force, the Coulomb force, the weak nuclear force and the strong nuclear force. Furthermore, there are many mechanisms and phenomena, which can result in a variety of forces and force characteristics like spring forces and contact forces. Some of these forces will be discussed in Sec. 6.13.

6.4 Procedure for determining the EoM

After obtaining a thorough understanding of kinematics, Newton's laws, mass and force, we are now in the position to *predict the motion* of point masses under the influence of forces, the essence of kinetics. The procedure to determine and solve the equations of motion can be subdivided in the following steps:

1. Sketch the point masses, massless mechanisms, force vectors and constraints
2. Choose and draw a suitable coordinate system (CS)
3. Determine the constraint equations
4. Draw the free-body diagram (FBD) for each object
5. Determine the equations of motion per object using Newton's second law
6. Solve the equations of motion (EoM) in the presence of constraints

We will discuss these steps in more detail in the following sections.

6.5 Sketch of the dynamic system

Before analysing the dynamics, it is important to make a sketch for a good overview of the dynamic system and the interactions between the different objects. Such a sketch, like shown in Fig. 6.3, should include the following elements:

- Constraining objects
- Massless mechanisms
- Objects with mass
- Forces vectors with points of action
- Labels
- Direction of the gravitational acceleration vector $\vec{g}$ (if present)
- Distances, angles and position vectors

- Coordinate systems and unit vectors

We will now discuss these different elements, starting with three different types of objects: constraining objects (like walls), objects with mass (like point masses) and massless mechanisms.

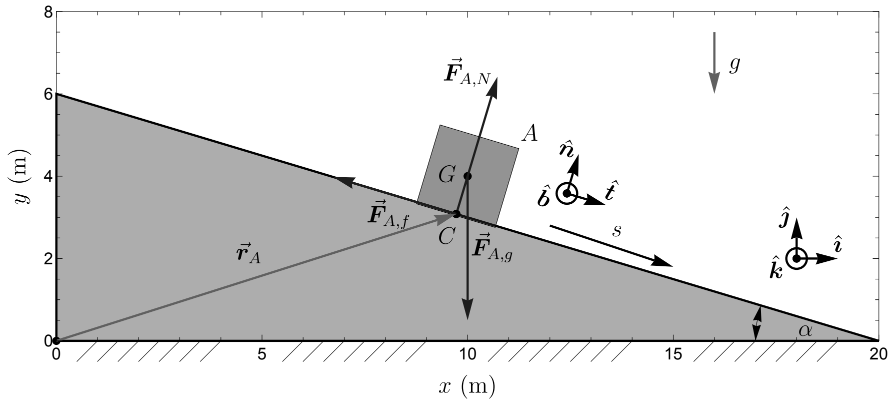


Figure 6.3: Sketch of a point mass A sliding along a slope.

6.5.1 Constraining objects

In the sketch, first draw all the *constraining objects*. These are the objects or structures that constrain the motion of other objects in the system. Constraining objects can either have a constant position, like walls, ground, ceiling and rails, but can also move, like elevators, drive shafts and carousels. In Fig. 6.3, the grey sloped surface is a constraining object. Since the time dependent motion of these constraining objects is fully known, Newton's second law doesn't have to be applied to them. This is normally also not possible since the mass of the constraining objects is usually not given. The effect of constraining objects can be described by constraint equations (Sec. 5.2).

6.5.2 Objects with mass

Secondly draw the masses at an arbitrarily chosen position and angle that satisfies the constraints.
Important: when choosing such an arbitrary position, make sure to *select a generic, non-trivial position*, for which all relevant forces are non-zero and have, if possible, non-zero components along the coordinate axes. For example, when drawing a pendulum, a mass suspended by a string, do not draw it in

the lowest position, since then there will be no horizontal component of the string force.

6.5.3 Massless mechanisms

Finally draw the *massless* mechanisms, these are mechanisms with zero or negligible mass, which can generate non-zero forces on point masses and constraining objects via mechanical connections. Examples include massless ropes, springs, rods, wheels, pulleys, gears and crankshafts and will be discussed in Sec. 6.13.6.

6.5.4 Force vectors

After all objects and mechanisms have been sketched, the force vectors can be sketched according to the guidelines in Sec. 3.2.4. We distinguish 3 types of force vectors:

1. *Force vectors at contact points.* Newton's third law holds for all forces, such that 2 force vectors should be drawn at each contact point between 2 objects and/or mechanisms: The force vector exerted by object 1 on object 2, and the (equal and opposite) force vector from object 2 on object 1. In Fig. 6.3 the forces $\vec{\boldsymbol{F}}_{A,N}$ and $\vec{\boldsymbol{F}}_{A,f}$ are force vectors that act on A at contact point C. Note that we often only draw the force vectors acting on objects for which we want to apply Newton's second law.
2. *Contactless forces acting at a distance.* Certain forces, like the gravitational force and the Coulomb force can act between 2 objects over a distance, without requiring mechanical contact. In Fig. 6.3 the gravitational force $\vec{\boldsymbol{F}}_{A,g}$ is a force with as point-of-action G, the center-of-mass of A.

6.5.5 Labelling objects and vectors

After drawing all objects and vectors, it is important to label them uniquely, since these labels link the sketch to the mathematics that you will use to solve the dynamics. Either letters, numbers or combinations of letters and numbers can be used as labels (see Sec. 10.7.2). For instance in Fig. 6.3 you have point mass A with mass m_A. Force vectors can be labelled with two subscripts, the first indicates the object on which the force acts and the second indicates the object or type of force which generates the force, e.g. $\vec{\boldsymbol{F}}_{A,N}$ is the normal force on A, $\vec{\boldsymbol{F}}_{A,g}$ the gravitational force and $\vec{\boldsymbol{F}}_{A,f}$ the friction force.

6.6 Choose and draw coordinate systems

After and while sketching the objects and forces, one or more relevant coordinate systems should be chosen. The choice of the coordinate system (CS) and its orientation is important, since a smart choice can significantly simplify the analysis. In most cases Cartesian coordinates are preferred. If you know the direction of a force (like the gravitational force or the contact force with the ground), it is often smart to align one of the coordinate axes with that force. Cylindrical coordinates are useful when objects rotate about a fixed rotation axis and t, n, b coordinates can be useful when point masses move along a constrained path, e.g. via a constraining rail object. In some cases a combination of coordinate systems can be used. In Fig. 6.3 one can use Cartesian coordinates and align the y-axis with the gravitational acceleration vector, however one can also choose to use natural t, n, b coordinates with path coordinate s. For demonstration, both are shown in the figure. It will become clear that in this situation t, n, b coordinates are easier to use[2]. After having chosen the CS it is drawn in the sketch with unit vectors according the guidelines from Sec. 3.3.

6.7 Determine the constraint equations

From the sketch and the coordinate system the constraint equations can be determined, like discussed in Sec. 5.2. In Fig. 6.3 we know that block A always touches the slope with angle α at point $A = C$. Since we have $\tan\alpha = \frac{y_A}{x_{A,0} - x_A}$, where $x_{A,0}$ is the coordinate at which the slope crosses the x-axis, we can determine the constraint equation and its time derivatives as:

$$y_A = (x_{A,0} - x_A)\tan\alpha \tag{6.6}$$

$$\dot{y}_A = -\dot{x}_A \tan\alpha \tag{6.7}$$

$$\ddot{y}_A = -\ddot{x}_A \tan\alpha \tag{6.8}$$

We can also use natural coordinates to express the constraint equations. It follows from Eq. (5.84) and Eq. (5.86) with radius of curvature $\rho = \infty$ because the slope is a straight line that the constraint equations then become:

$$v_{A,n} = 0 \tag{6.9}$$

$$a_{A,n} = 0 \tag{6.10}$$

[2]Drawing the x-axis parallel to the slope would also have simplified the problem, so also consider rotating your coordinate system to facilitate calculations.

6.8 Drawing the free-body diagram

The free-body diagram (FBD) is a drawing of an object (=body) or a set of objects, isolated from their surroundings and all force vectors and moment vectors that act on it. Its main purpose is to determine the sum of the force vectors $\sum \vec{\boldsymbol{F}}$. The word free indicates that the object is cut free from surrounding objects, and all effects of the surroundings are represented by force and moment vectors. A key purpose of an FBD is to define the difference between internal and external forces, since all objects drawn in the FBD can be considered to be *internal* and their forces should not be drawn, whereas all drawn forces are *external* forces generated by objects that are *external* and not drawn. Although we will focus on the FBD for single point masses in this chapter, we will later also consider FBDs of objects and rigid bodies consisting of many point masses.

If a good sketch has been made in the previous steps, drawing the FBD is straightforward: draw the object, and draw all force vectors that act on the object just like they were drawn in the sketch. An example is given in Fig. 6.4. Also provide the unit vectors, which are used to obtain the force components by projection.

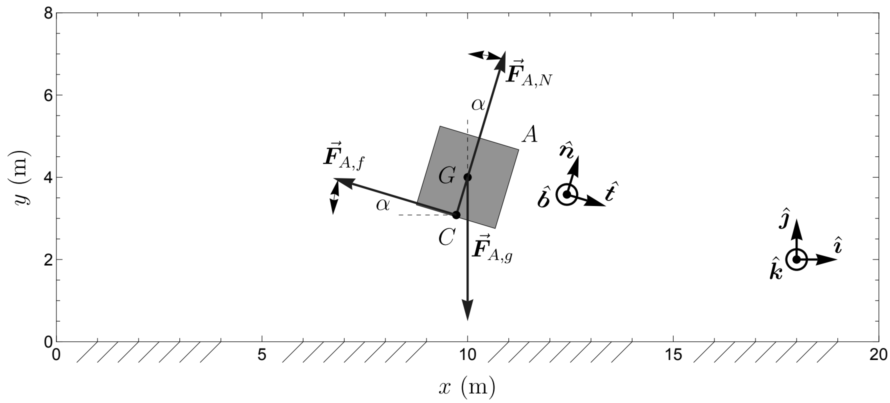


Figure 6.4: Free-body diagram of the block shown in Fig. 6.3.

6.8.1 Drawing the FBD

To draw the FBD we follow the methodology as described in Vallery and Schwab [9]. This description already includes the elements in the FBD needed for rigid bodies in the presence of moments that will be introduced in Ch. 10.

- Draw the system in a free state, i.e. "cut" the system at convenient locations. Don't draw objects that do not belong to the system. Draw outlined shapes of the separate pieces belonging to the system. Each cut at a contact point can introduce new external action-reaction forces at the system boundaries. Some helpful guidelines:

  a) Always draw the system in a generic state. So, for example, if you draw the FBD of a pendulum, do not draw it in the vertical position, since in that position the horizontal component of the force in the rope is zero, while it is not always zero.

  b) Whenever possible, choose your system boundaries such that you expose only action-reaction forces that you are actually interested in calculating, or that are easily determined from the information you have. Otherwise, extra equations and unknowns are introduced, complicating calculations.
- Draw a 3D coordinate system with a clearly defined position for the origin O. Check if rotating and/or translating the coordinate system can make things easier, and if Cartesian or cylindrical coordinate systems makes the analysis simplest. Consider drawing the three unit vectors. If the coordinate system is already clear from the sketch it may be omitted in the FBD.
- Some rules for drawing forces and moments in a FBD:

  a) Do not show internal forces or moments.

  b) If a connection prevents movement or rotation in a particular direction, then forces/moments are drawn to represent that restriction.

  c) If a rigid segment is split in two, the forces and moments acting on the two segments in the separated FBDs are equal in magnitude and opposite in direction.

  d) Forces acting on a rigid body may be shifted along their lines of action (see sliding vectors in Ch. 3).

  e) Couple moments acting on a rigid body may be placed anywhere (they are free vectors).

  f) Only draw moment vectors if the forces that generate them are not drawn to prevent double counting of forces or moments.

  g) Projections of forces on the coordinate axes can be drawn in a FBD, but should be uniquely labelled as such (e.g. $F_{g,x}$ and $F_{g,y}$), and use dashed arrows to distinguish them from the original force.

  h) Draw the arrows of the force vector in the expected direction of the force.
- Indicate and label all known and unknown external force vectors and moment vectors that act on the system uniquely, and place dots at the

correct points of action and reference points. Decide on the preferred vector notation method Sec. 3.2.4.
- Label relevant dimensions, distances, angles and relative position vectors.

The main goal of these guidelines is that an FBD should include all relevant information needed to correctly establish the sums of forces and moments that will later be used to determine the equations of motion.

6.8.2 Common errors in drawing FBDs

To help you generate proper FBDs, we list errors that often occur in drawing FBDs.

- Drawing internal forces.
- Drawing forces generated *by* the free body (e.g. on the constraints). Only the external forces acting *on* the free body should be drawn.
- Drawing the same force twice.
  Both drawing the force and its moment (can lead to double counting).
  Drawing both the vector and its projection on the axes without clearly differentiating them, e.g. with dashed lines (can lead to double counting).
- Drawing velocity or acceleration vectors in an FBD without clearly distinguishing them from the forces.
- Drawing objects that do not belong to the free body in the FBD.
- Forgetting to draw forces generated by constraints on the body.
- Forgetting gravity or other forces.
- Forgetting to draw or indicate a coordinate system or unit vector directions.
- Drawing the FBD in a non-generic (=trivial) state or equilibrium position (see previous subsection), such that certain force components or angles become zero, while they are not always zero.
- Using scalar unsigned magnitude notation (e.g. F_A see Ch. 3), for a force of which the direction is unknown or time-dependent.
- Drawing resultant or sums of vectors like $\vec{\boldsymbol{F}}_{\text{tot}}$ (leads to double counting). Each individual physical force vector needs to be drawn as a separate vector.
- Already using Newton's laws to determine the value of forces, e.g. writing $mg\hat{\boldsymbol{\jmath}}$ for the normal force, instead of keeping it as an unknown and determining it from Newton's laws.
- Drawing a vector $m\vec{\boldsymbol{a}}$. Acceleration is a result of force, but is not the force itself. An FBD should only contain the physical forces acting on

the free body.

- Drawing virtual or pseudo force vectors, based on assumed motions like the centrifugal (or centripetal) force in rotations. Only forces resulting from physical interactions between objects can be drawn in the FBD, since it is the purpose of the FBD to determine the sum of forces and use that to determine the motion.

6.9 Force expressions and projections

Now we express the forces drawn in the FBD in the form of equations. For the forces in Fig. 6.4, assuming a kinetic friction coefficient μ_k during the sliding motion we get:

$$\vec{\boldsymbol{F}}_{A,g} = -m_A g \hat{\boldsymbol{\jmath}} \tag{6.11}$$

$$\vec{\boldsymbol{F}}_{A,N} = F_N \hat{\boldsymbol{n}} \tag{6.12}$$

$$\vec{\boldsymbol{F}}_{A,f} = -\mu_k F_N \hat{\boldsymbol{t}} \tag{6.13}$$

Projecting the force vectors

After having drawn the object and force vectors in the FBD, we need their scalar components to evaluate Newton's second law along each axis. We thus project the forces on the coordinate system (CS) that was selected in the previous section and was drawn in the sketch and/or FBD. This is done using the method described in Sec. 3.2.5, and shown in Fig. 6.5. For example we get the x component of the vector $\vec{\boldsymbol{F}}_{A,f}$ using $F_{A,f,x} = \vec{\boldsymbol{F}}_{A,f} \cdot \hat{\boldsymbol{\imath}}$.

As projected components of $\vec{\boldsymbol{F}}_g$ we obtain:

$$F_{A,g,t} = m_A g \sin\alpha \tag{6.14}$$

$$F_{A,g,n} = -m_A g \cos\alpha \tag{6.15}$$

6.10 Obtaining the equations of motion

After thorough preparation, we are now in the position to fully evaluate Newton's second law to obtain the equations of motion using the following steps.

1. Add the force components from the FBD along each of the axes.
2. Apply $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ along each of the axes.
3. Express the components of the acceleration vector in terms of the position coordinates and their time derivatives using kinematics (see Sec. 5.9).

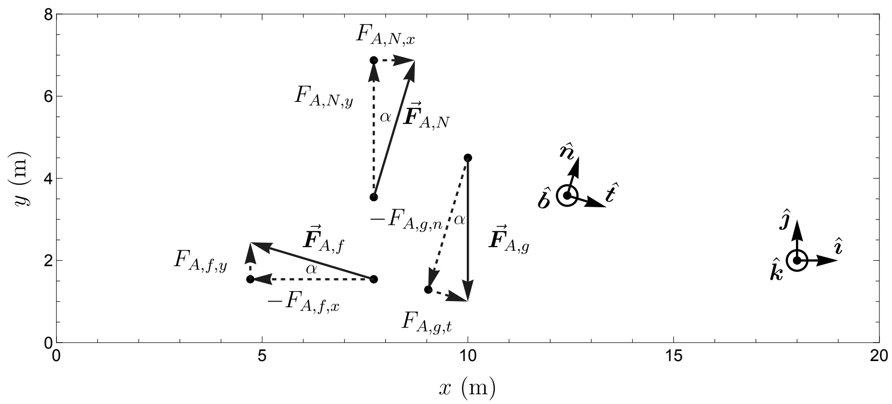


Figure 6.5: Projecting the forces in Fig. 6.4 on the coordinate axes. Note that minus signs are put in front of the labels of dashed arrows that point in the negative axis direction to conform to the conventions of drawing vectors with a scalar label in Sec. 3.2.2.

4. Simplify the EoM by using the constraint equations and combining the scalar equations along the axes directions.

S **Example 6.1** As an example of this procedure we apply it to obtain the EoM of the block A in Fig. 6.4 using natural coordinates, and the projected forces from the previous section. We obtain three scalar equations of motion.

$$\sum F_{A,t} = m_A g \sin\alpha - \mu_k F_N = m_A a_{A,t} = m_A \ddot{s}_A \tag{6.16}$$

$$\sum F_{A,n} = F_N - m_A g \cos\alpha = m_A a_{A,n} \underset{\text{Using Eq. (6.10)}}{=} 0 \tag{6.17}$$

$$\sum F_{A,b} = 0 = m_A a_{A,b} = m_A \ddot{z}_A, \tag{6.18}$$

where we used Eq. (5.85) and Eq. (5.86) to obtain the acceleration components in t, n, b coordinates. We see that the sum of forces in the $\hat{\boldsymbol{b}}$ direction is zero, such that $\ddot{z}_A$ is zero. We uses the constraint equation Eq. (6.10) to find that $a_{A,n} = 0$, which implies that the sum of the forces in that direction is also zero. From that condition we find the equation for the normal force[a] to be $F_N = m_A g \cos\alpha$. By substituting this equation into Eq. (6.16) we obtain the EoM along the $\hat{\boldsymbol{t}}$ direction:

$$m_A \ddot{s}_A = m_A g \sin\alpha - \mu_k F_N \tag{6.19}$$

$$m_A \ddot{s}_A = m_A g \sin\alpha - \mu_k m_A g \cos\alpha \tag{6.20}$$

$$\ddot{s}_A = g(\sin\alpha - \mu_k \cos\alpha) \equiv a_0, \tag{6.21}$$

where we define a_0 as the constant acceleration along the path coordinate. The same result could have been obtained by using Cartesian coordinates, but this would have been more complicated mathematically.

[a]It is very important that in dynamics the normal forces are determined from the combination of EoM and constraint equations.

6.11 Equations of motion in kinetics

6.11.1 Solving the equations of motion

The goal of solving the equations of motion is to obtain the position vectors $\vec{\boldsymbol{r}}_i(t)$ of all objects i at all times t for given initial conditions. Solving these equations is part of kinematics and was already discussed in Ch. 5. We will briefly repeat the procedure here for the example from the previous section.

Since Newton's second law gives us an expression for the acceleration $\vec{\boldsymbol{a}}(t)$, the main challenge in solving the EoMs is to determine the time dependent position vector $\vec{\boldsymbol{r}}(t)$ of an object from the acceleration $\vec{\boldsymbol{a}}(t)$. Often this can be done using the kinematic integration techniques that we have discussed in the previous chapter. In some cases more complex differential equations, that cannot be solved by integration appear, as will be discussed in Ch. 13.

As an example we consider the block A in Fig. 6.4. We know from the constraint equations that $a_{A,n} = 0$ and $a_{A,b} = 0$. So, we only need to determine the motion $s_A(t)$ along the path coordinate. For a certain initial condition $s_A(0) = s_0$ and $\dot{s}_A(0) = v_0$, we integrate the equation of motion twice to obtain the motion as follows 6.21:

$$v_A(t) = v_0 + \int_0^t a_0 \mathrm{d}t = v_0 + a_0 t \tag{6.22}$$

$$s_A(t) = s_0 + \int_0^t (v_0 + a_0 t)\mathrm{d}t = \frac{1}{2}a_0 t^2 + v_0 t, \tag{6.23}$$

where a_0 was obtained from the EoM in Eq. (6.21).

6.11.2 Determining forces from the EoM

Besides using the EoM to determine the motion from known forces, it is also possible to substitute knowledge of the motion into the equation of motion, to determine unknown forces using the EoM. Essentially, this method follows directly from Newton's second law $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, where the sum of forces can be determined if the mass m and acceleration $\vec{\boldsymbol{a}}$ are known. Note that since the sum of forces is obtained, if more than one force acts on the mass, this equation allows only obtaining one unknown force vector or 3 scalar force components.

An example of how forces can be determined from knowledge of the acceleration was already given in Example 6.1, where we determined in Eq. (6.17) the contact force F_N by combining the knowledge that the acceleration component perpendicular to the slope was zero ($a_{A,n} = 0$) and the knowledge of the mass and gravitational force. This is a powerful way to study and quantify forces.

6.12 Relative motion and inertial reference frames

We will now discuss what happens if two persons (observers), that move relative to each other, both apply Newton's laws. We first introduce the concept of a reference frame.

Concept. *Reference frame*
The points that are chosen to have zero velocity and acceleration are called the reference frame of an observer. Velocities and accelerations of objects can be measured with respect to the reference frame by choosing a coordinate system that is fixed with respect to the reference frame.

To illustrate the situation of two observers using different reference frames, consider two boats A and B that move with position vectors $\vec{\boldsymbol{r}}_A(t)$ and $\vec{\boldsymbol{r}}_B(t)$ relative to an origin O that is fixed to the quay reference system (see Fig. 6.6). Observer O stands on the quay and uses an x, y, z coordinate system that is fixed in the quay reference frame. A second observer A on boat A uses coordinate system x', y', z' with origin A, that is fixed to boat A's reference frame. While the position of boat B in O's reference system is $\vec{\boldsymbol{r}}_B(t)$, in A's reference system it is given by:

$$\vec{\boldsymbol{r}}'_B(t) = \vec{\boldsymbol{r}}_{B/A}(t) = \vec{\boldsymbol{r}}_B(t) - \vec{\boldsymbol{r}}_A(t) \tag{6.24}$$

By taking the time derivatives of Eq. (6.24), the velocity and acceleration of boat B in A's reference frame can be determined:

$$\begin{aligned} \vec{\boldsymbol{v}}'_B(t) &= \vec{\boldsymbol{v}}_{B/A}(t) = \vec{\boldsymbol{v}}_B(t) - \vec{\boldsymbol{v}}_A(t) && (6.25) \\ \vec{\boldsymbol{a}}'_B(t) &= \vec{\boldsymbol{a}}_{B/A}(t) = \vec{\boldsymbol{a}}_B(t) - \vec{\boldsymbol{a}}_A(t) && (6.26) \end{aligned}$$

So observer O will conclude that the acceleration of boat B is $\vec{\boldsymbol{a}}_B$, and observer A will conclude that the acceleration of boat B is $\vec{\boldsymbol{a}}'_B = \vec{\boldsymbol{a}}_{B/A}$. When both observers apply Newton's laws to boat B observer O will find that the total force acting on boat B is $\vec{\boldsymbol{F}}_B = \vec{\boldsymbol{a}}_B/m_B$ and observer A finds that the force is

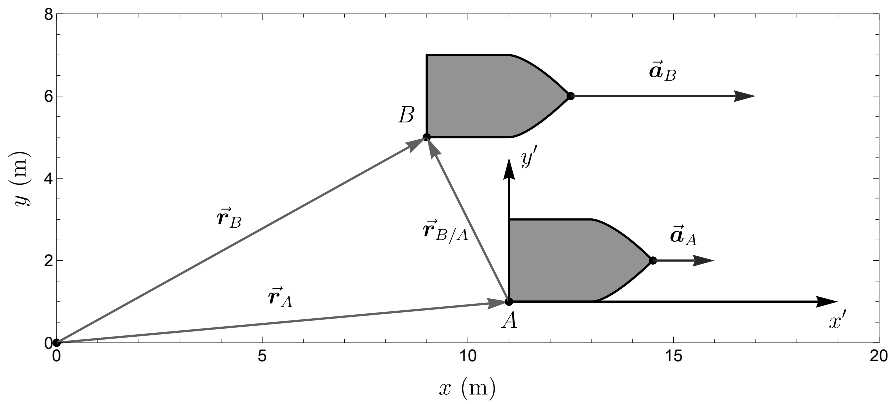


Figure 6.6: Newton's laws do not hold for an accelerating reference frame ($\vec{a}_A \neq 0$) because one finds different acceleration vectors for boat B using such a system: $\vec{a}_B \neq \vec{a}'_B = \vec{a}_{B/A}$. Such a frame fixed to an accelerating boat A is therefore not an IRF.

$\vec{F}_B = \vec{a}'_B/m_B$. Of course the force acting on boat B cannot depend on the acceleration of boat A. Both observers should therefore find the same value for $\vec{F}_B$, which is only true if $\vec{a}_B = \vec{a}'_B = \vec{a}_{B/A}$. This only holds if the acceleration of boat A in O's reference system is zero: $\vec{a}_A = \vec{0}$.

Inertial reference frames and pseudo forces

From this argumentation we conclude that Newton's second law cannot be valid in all reference frames, but only in special reference frames that we call inertial reference frames.

Concept. *Inertial reference frame (IRF)*
An inertial reference frame is a reference frame in which Newton's second law is valid. It is a reference frame which does not accelerate or rotate substantially (relative to distant stars).

One might ask, how can observer A check if she is in an IRF? She can do so by testing if Newton's second law holds in her reference frame, by positioning a point mass at a fixed position in the boat without any forces acting on it. If the mass accelerates in the absence of forces, the observer knows that she is not in an IRF. From the observed acceleration, observer A might conclude, based on Newton's second law, that a kind of force is acting on the mass. However, since there are no interactions this force is not a real force, instead it is a fictitious force that is called a *pseudo force*, since it is only observed because the experiment is performed in an accelerating reference frame instead of an IRF. Examples of such pseudo forces are the effect pulling you forward when a car brakes and the centrifugal 'force' that pulls you radially when a car takes a turn. Unless explicitly mentioned, for the rest of this textbook we only deal with dynamics in inertial reference frames (IRFs).

6.13 Forces and constraints

In this section we discuss the properties of some of the most relevant forces and constraints that can occur in dynamics, as have been experimentally approximately determined using Newton's laws.

6.13.1 Gravitational force

The gravitational force is important, because it acts on every point mass, and because it is the most prominent force that acts over a distance. In this textbook we focus on the dynamics of objects on or near the surface of the earth, and therefore we can often assume a constant distance to the earth's centre. In this case the gravitational force vector on a mass m_i can be described by

$$\vec{\boldsymbol{F}}_{i,g} = m_i \vec{\boldsymbol{g}}, \tag{6.27}$$

where $\vec{\boldsymbol{g}}$ is the gravitational acceleration vector with magnitude $g = |\vec{\boldsymbol{g}}| \approx 9.8\ \mathrm{m/s^2}$ that points downward in the direction of the centre of the earth.

6.13.2 Contact force

At a point C where the surfaces of two objects touch, the atoms in the objects can generate a *contact force* vector $\vec{\boldsymbol{F}}_C$. One can define a *contact plane* through C that is tangential to the surface of the objects. Then, like shown in Fig. 6.3, the component of the contact force $\vec{\boldsymbol{F}}_C$ perpendicular to this plane is called the *normal force* $\vec{\boldsymbol{F}}_N$, and the component parallel to the contact plane is the called the friction force $\vec{\boldsymbol{F}}_f$. So, it always holds that $\vec{\boldsymbol{F}}_C = \vec{\boldsymbol{F}}_N + \vec{\boldsymbol{F}}_f$, and it is allowed to draw and treat the contact force as a single vector $\vec{\boldsymbol{F}}_C$. However, since the expressions for $\vec{\boldsymbol{F}}_N$ and $\vec{\boldsymbol{F}}_f$ are different, it can be useful to split them. Let us now discuss the normal forces and friction forces separately.

6.13.3 Normal force

Normal forces are the result of repulsive Coulomb forces and quantum mechanical effects. By definition, the normal force vector $\vec{\boldsymbol{F}}_N$ points perpendicular to the contact plane. Normal forces can be calculated by combining Newton's laws with constraint equations. If the contact surface is flat, the constraint equations for the motion components normal to the surface simplify to $v_n = 0$ and $a_n = 0$, otherwise one needs to use Eq. (5.86). An example of how to determine the normal force F_N from the EoM was given in Eq. (6.10) and Eq. (6.17), where the movement of block A was constrained by the slope.

6.13.4 Friction force

The friction force is the component of the contact force tangential to the contact plane. We distinguish static or kinetic friction forces, which are characterised by a static μ_s and kinetic μ_k friction coefficient with $\mu_s \geq \mu_k > 0$. We now consider the friction force between two points A and B on different objects that are in contact.

Static friction

The static friction acting on point A is best described by this constraint equation:

$$\begin{cases} a_{A/B,t} = 0 & \text{if } |\vec{\boldsymbol{F}}_{A,fs}| \leq \mu_s |F_{A,N}| \text{ and } \vec{\boldsymbol{v}}_{A/B} = \vec{\boldsymbol{0}} \\ & \text{otherwise use kinetic friction} \end{cases} \tag{6.28}$$

The first condition for static friction is that the points A and B are not moving relative to each other $\vec{\boldsymbol{v}}_{A/B} = 0$. In that case Eq. (6.28) states that the tangential component of acceleration of point A is zero ($a_{A/B,t} = 0$) as long as the friction force $|\vec{\boldsymbol{F}}_{A,fs}|$ needed to satisfy that constraint equation does not exceed the maximum static friction force $F_{A,fs,\max} = \mu_s |F_{A,N}|$, where $F_{A,N}$ is the normal force on A. In practice the EoM in the presence of the constraint $a_{A/B,t} = 0$ needs to be solved to determine the friction force $\vec{\boldsymbol{F}}_{A,fs}$ after which it is checked if the static friction condition holds. If it does not hold, point A will accelerate and kinetic friction can be used.

Kinetic friction

When the points A and B slide or slip along each other, a kinetic friction force acts on A. The kinetic friction force vector $\vec{\boldsymbol{F}}_{A,fk}$ is tangential to the plane of contact and points opposite to the direction of the relative velocity vector:

$$\vec{\boldsymbol{F}}_{A,fk} = \begin{cases} -\mu_k |\vec{\boldsymbol{F}}_{A,N}| \hat{\boldsymbol{v}}_{A/B} & \text{if } \vec{\boldsymbol{v}}_{A/B} \neq 0 \text{ or } |\vec{\boldsymbol{F}}_{A,fs}| > \mu_s |F_{A,N}| \\ & \text{otherwise use static friction} \end{cases} \tag{6.29}$$

Solving problems with friction

If it is not known whether one should deal with static or kinetic friction, and/or if the direction of the velocity is not known in a kinetic friction situation, the following procedure can be used to solve the problem step by step:

1. Determine the normal contact force F_N using constraint equations and the EoMs, like in Eq. (6.17).
2. Assume that the relative velocity is zero $\vec{\boldsymbol{v}}_{A/B} = \vec{\boldsymbol{0}}$ and use the constraint equation $a_{A/B,t} = 0$ to determine the static friction force $\vec{\boldsymbol{F}}_{A,fs}$ from the EoMs.
3. Check if $|\vec{\boldsymbol{F}}_{A,fs}| \leq \mu_s F_N$. If that is true, the static friction assumption is correct.
4. Otherwise use kinetic friction. Guess a direction for the velocity $\hat{\boldsymbol{v}}_{A/B}$ and use it to determine $\vec{\boldsymbol{F}}_{A,fk}$ using Eq. (6.29).
5. Then solve the EoMs to determine if the direction of the velocity $\vec{\boldsymbol{v}}_{A/B}$ matches the guessed direction of $\hat{\boldsymbol{v}}_{A/B}$.
6. If not, repeat from step 4 for the other direction of $\hat{\boldsymbol{v}}_{A/B}$.

For friction of a rolling wheel with the ground two conditions can occur. If there is *no slip*, only static friction is present and $a_{A/B,t} = 0$. While in the case of *slip* or sliding, one deals with kinetic friction forces. Gearwheels always behave according to the *no slip* condition, since their teeth prevent slipping. Finally, a special situation are so-called *frictionless* contact problems, which have $\mu_s = \mu_k = 0$. For these cases the friction force is always zero $\vec{\boldsymbol{F}}_f = \vec{\boldsymbol{0}}$, and the contact force is always normal to the contact plane.

6.13.5 Constraint equations on points

Some examples of common constraint equations that limit the motion of a point A in 3, 2 or 1 dimensions are:

- A joint that fixes a point A to a certain other point P, fixing it along all 3 coordinate directions:
  $\vec{\boldsymbol{r}}_A = \vec{\boldsymbol{r}}_P$

- A point A is constrained to move along a slider joint, collar or rail with path curve $\vec{\boldsymbol{r}}_s(s)$, constraining it along 2 coordinate axes:
  $\vec{\boldsymbol{r}}_A = \vec{\boldsymbol{r}}_s(s)$.
- Motion over a surface, for instance because the gravitational force ensures object A does not move in the z direction, constraining motion along 1 coordinate direction:
  $z_A(t) = 0$
- Angular constraints can fix certain rotation angles, e.g. $\phi(t) = \phi_0$, they can result in moments on the constrained object.

The contact forces and moments resulting from these constraint equations can be analyzed by combining them with the EoMs.

6.13.6 Dynamics of massless mechanisms

When a mechanism m has zero mass, so $m_m = 0$, application of Newton's second law tells us that the sum of external forces that act on the mechanism is zero, since otherwise its acceleration would be infinite:

$$\sum \vec{\boldsymbol{F}}_{\text{ext}} = m_m \vec{\boldsymbol{a}}_m = \vec{\boldsymbol{0}} \tag{6.30}$$

This means that the sum of the forces on the mechanism is always zero, just like in statics. Therefore you can use all methods you have learned in statics courses to determine the positions and forces of the massless elements in the mechanism. Unless it is explicitly mentioned they have mass, you may assume all mechanisms in this textbook to be massless.

6.13.7 Two-force members

A massless mechanism on which only two external forces $\vec{\boldsymbol{F}}_A$ and $\vec{\boldsymbol{F}}_B$ are acting is called a two-force member. Then we get from Eq. (6.30) that $\vec{\boldsymbol{F}}_A + \vec{\boldsymbol{F}}_B = \vec{\boldsymbol{0}}$ and:

$$\vec{\boldsymbol{F}}_A = -\vec{\boldsymbol{F}}_B \tag{6.31}$$

We see from Eq. (6.30) that because the mass of the mechanism is zero ($m_m = 0$), its acceleration $\vec{\boldsymbol{a}}_m$ can take any value, which means that the mechanism can translate extremely fast, such that it will instantaneously assume a position that ensures $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$. Note that, as will become clearer later, the same argumentation holds for rotations, such that the mechanism also instantaneously assumes a state where the sum of external moments is zero $\sum \vec{\boldsymbol{M}} = \vec{\boldsymbol{0}}$, which is only possible if the two forces $\vec{\boldsymbol{F}}_A$ and $\vec{\boldsymbol{F}}_B$ are collinear. We will now discuss a few important massless mechanisms, show in Fig. 6.7.

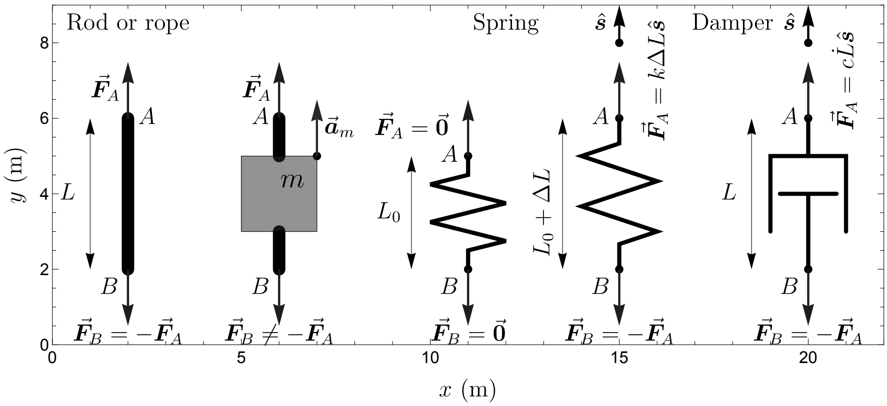


Figure 6.7: FBDs of various mechanisms: a rod, a rope with mass, a relaxed spring with length L_0, a spring with length $L_0 + \Delta L$ and a damper.

6.13.8 Rods and ropes

Probably the simplest two-force members are rods and ropes. Since these mechanisms have a fixed length L their operation is governed by this constraint equation:

$$|\vec{\boldsymbol{r}}_A - \vec{\boldsymbol{r}}_B| = L \tag{6.32}$$

Ropes are flexible, therefore they can only handle tensile ('pulling') forces that are parallel to the rope. The absolute value of those force vectors is called the tension in the rope and Eq. (6.32) only holds if the rope is experiencing tensile force. If ropes are bent (e.g. by pulleys) they can be dealt with as discussed in 5.2.4. Since rods are rigid, they can handle also compressive forces and can transfer moments.

As shown in second FBD in Fig. 6.7 the sum of the forces on a rope is not zero anymore ($\vec{\boldsymbol{F}}_A + \vec{\boldsymbol{F}}_B \neq 0$) if there is mass connected to the rope, because part of the force $\vec{\boldsymbol{F}}_A$ is needed to accelerate the mass. So, be careful, $\vec{\boldsymbol{F}}_A = -\vec{\boldsymbol{F}}_B$ can only be applied when no masses are accelerated by the forces.

6.13.9 Spring

A linear spring is a mechanism whose length extends by a distance ΔL that is proportional to the external pulling force (see Fig. 6.7). When there is no external force ($\vec{\boldsymbol{F}}_A = \vec{\boldsymbol{F}}_B = \vec{\boldsymbol{0}}$), it has a *relaxed length* $L = L_0$, and when a force is applied its length increases by a distance $\Delta L = L - L_0$:

$$\vec{\boldsymbol{F}}_k = k\Delta L\hat{\boldsymbol{s}} \tag{6.33}$$

Here k is the stiffness or spring constant of the spring, with unit N/m, which normally has a positive value. The direction of the force can be indicated using a unit vector $\hat{\boldsymbol{s}}$ that points outward parallel to the spring at point A.

6.13.10 Damper

A linear damper is a mechanism whose length changes at a rate that is proportional to the externally applied pulling force (see Fig. 6.7).

$$\vec{\boldsymbol{F}}_c = c\frac{\mathrm{d}L}{\mathrm{d}t}\hat{\boldsymbol{s}} \tag{6.34}$$

where c is the damping constant (unit N·s/m), which is always positive. If point B is fixed, the velocity of point A is in the same direction as the applied force vector.

It is important to note that besides extending, springs and dampers can also be compressed, such that the distance ΔL and time derivative $\dot{L}$ become negative and the forces change sign. Secondly we note that we have discussed in Fig. 6.7 the effect of external forces $\vec{\boldsymbol{F}}_A$ and $\vec{\boldsymbol{F}}_B$ *generated on* the mechanisms. But if one e.g. connects a point mass m at point A to a spring, the force *generated by* the mechanism is of opposite sign $\vec{\boldsymbol{F}}_{m,k} = -k\Delta L\hat{\boldsymbol{s}}$, where the minus sign arises from Newton's third law.

6.13.11 Other forces

There are many other types of forces, like thermodynamic, aerodynamic, fluid dynamic, electromagnetic, electrostatic and material forces. It carries too far here to discuss them all, and the methodology discussed in this chapter can be applied for all types of forces once the functional form is known.

6.14 Summary

In this chapter we have formulated a complete methodology to derive the kinetics of systems of one or more point masses under the influence of forces and constraints generated by constraining objects and massless mechanisms. Here we have made use of the kinematics from the previous chapter, introduced the important concepts force and mass, and discussed the procedure of sketching the problem and the FBDs to obtain and solve the EoMs. Let us summarise the main topics and equations in this chapter:

- **Mass and force**
  - Mass and force are defined by Newton's laws
  - $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$
  - $\vec{\boldsymbol{F}}_{\text{action}} = -\vec{\boldsymbol{F}}_{\text{reaction}}$
- **Procedure for solving the EoMs**
  - Sketch, CS, FBD, projecting forces, determining the EoM
  - Solving the EoM using kinematics and constraints
  - Relative motion and IRF
- **Forces, mechanisms and constraints**
  - Gravity: $\vec{\boldsymbol{F}}_g = m\vec{\boldsymbol{g}}$
  - Friction: Eqs. (6.28,6.29)
  - Massless mechanisms: $\sum \vec{\boldsymbol{F}}_{\text{ext}} = \vec{\boldsymbol{0}}$
  - Rods and ropes: $|\vec{\boldsymbol{r}}_A - \vec{\boldsymbol{r}}_B| = L$
  - Spring: $\vec{\boldsymbol{F}}_k = k\Delta L\hat{\boldsymbol{s}}$
  - Damper: $\vec{\boldsymbol{F}}_c = c\frac{\mathrm{d}L}{\mathrm{d}t}\hat{\boldsymbol{s}}$

Although all kinetic problems can be solved using the general method outlined in this chapter, certain problems, in particular those where the forces only depend on position, are easier to solve using the concepts of work and energy, which will be discussed in the next chapter. If the forces only depend on time, or have a very short duration, the principle of impulse and momentum is useful, which will be discussed in chapter 8.

7 Work and Energy

In the previous chapters we have outlined a methodology to fully analyse the dynamics of point masses from their kinetics and kinematics. However, in certain situations information about the motion can be obtained via an easier route, using the methods of work and energy. The methods of work and energy are particularly useful if:

- The forces depend on position and not explicitly on time
- One only needs to determine changes in speed
- It is not needed to fully determine the motion $\vec{\boldsymbol{r}}(t)$

7.1 Principle of work and energy

We start by introducing the most important principle of this chapter.

Concept. *Principle of work and energy*
The principle of work and energy states that the work W done by a force $\vec{\boldsymbol{F}}_{ij}$ on a point mass i, while it moves along a path curve from position s_1 to position s_2, is equal to the change in its kinetic energy T_i.

$$W_{ij,s_1\to s_2} = \Delta T_{s_1\to s_2}, \tag{7.1}$$

where $\Delta T_{s_1\to s_2} = T_i(s_2) - T_i(s_1)$ is the change in kinetic energy. To use the principle in Eq. (7.1) we first need to define the work $W_{ij,s_1\to s_2}$ done by a force and the kinetic energy T_i of the point mass.

7.1.1 Work

Concept. *Work*
The work W_{ij} performed ***by*** *a force $\vec{\boldsymbol{F}}_{ij}$* ***on*** *a point mass i, while it moves from path coordinate s_1 to s_2 along a path curve $\vec{\boldsymbol{r}}_s(s)$ is defined as:*

$$W_{ij,s_1\to s_2} \equiv \int_{\vec{\boldsymbol{r}}_s(s_1)}^{\vec{\boldsymbol{r}}_s(s_2)} \vec{\boldsymbol{F}}_{ij}\cdot \mathrm{d}\vec{\boldsymbol{r}} \tag{7.2}$$

This expression can be simplified by using that $\mathrm{d}\vec{\boldsymbol{r}} = \hat{\boldsymbol{s}}\mathrm{d}s$ (see Eq. (5.21)). We can project the force vector on the path curve, obtaining its tangential component using:

$$F_{ij,t} = \vec{\boldsymbol{F}}_{ij} \cdot \hat{\boldsymbol{s}} \tag{7.3}$$

Thus we obtain the following scalar expression for the work:

$$W_{ij,s_1 \to s_2} = \int_{s_1}^{s_2} F_{ij,t}\mathrm{d}s \tag{7.4}$$

7.1.2 Kinetic energy

Concept. *Kinetic energy of a point mass*
The kinetic energy of a point mass i is defined as one half its mass times its speed squared ($v_i^2 = |\vec{\boldsymbol{v}}_i|^2$), and is indicated by the letter T.

$$T_i \equiv \frac{1}{2} m_i v_i^2 \tag{7.5}$$

S **Example 7.1** As an example of the principle of work and energy we determine the change in kinetic energy of a ball that is launched at an initial velocity $\vec{\boldsymbol{v}}_0 = v_{x,0}\hat{\boldsymbol{\imath}} + v_{y,0}\hat{\boldsymbol{\jmath}}$ and experiences constant force $\vec{\boldsymbol{F}}_g = -mg\hat{\boldsymbol{\jmath}}$ of gravity.

The trajectory of the ball is shown in Fig. 7.1. We first determine the work done by the force on the ball using Eq. (7.2). By using that $\mathrm{d}\vec{\boldsymbol{r}} = \mathrm{d}x\hat{\boldsymbol{\imath}} + \mathrm{d}y\hat{\boldsymbol{\jmath}}$, such that $\vec{\boldsymbol{F}}_g \cdot \mathrm{d}\vec{\boldsymbol{r}} = -mg\mathrm{d}y$ we find:

$$W_{s_1 \to s_2} = \int_{y_1}^{y_2} -mg\mathrm{d}y = -mg(y_2 - y_1) \tag{7.6}$$

So, as shown in Fig. 7.2, the work done by the gravitational energy is negative and has the same parabolic shape as the trajectory of the ball shown in Fig. 7.1. The work is negative because the tangential component of the force initially reduces the speed. After the ball has passed its highest point, the work increases again, because the tangential component of gravity is in the same direction as the velocity vector.

From the principle of work and energy Eq. (7.1) we see that the change in kinetic energy ΔT_{12} equals $W_{s_1 \to s_2}$, and follows the same curve in Fig. 7.2. Note that the work could also have been determined by projecting the force on the path curve using angle α and using the scalar equation Eq. (7.4) but this would have been more difficult.

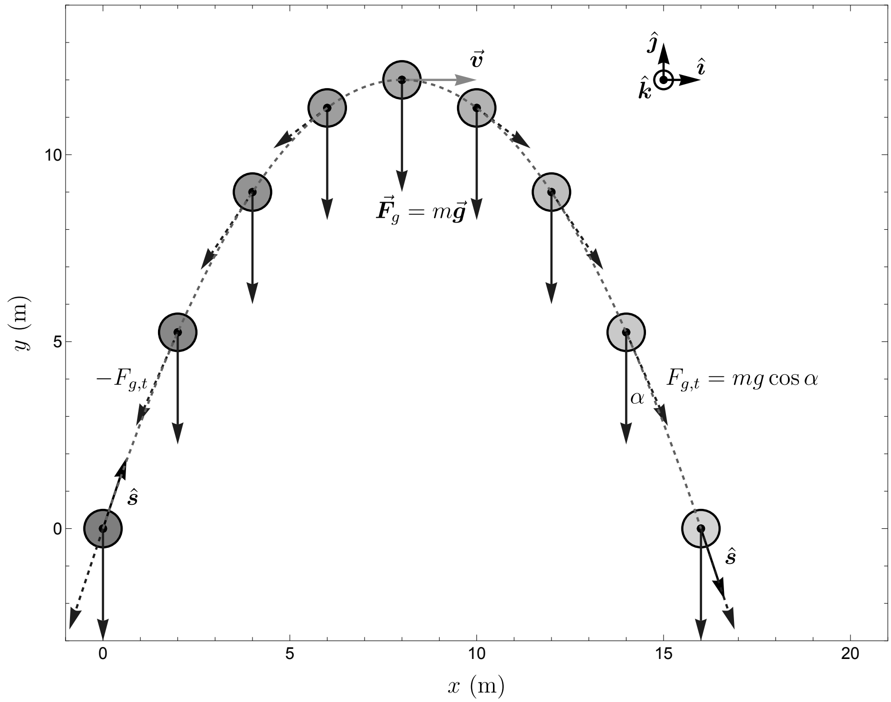


Figure 7.1: Trajectory of a ball experiencing a constant gravitational force of $\vec{\boldsymbol{F}}_g = -0.5N\hat{\boldsymbol{\jmath}}$.

7.1.3 Derivation of the principle

Here we present the derivation of the principle of work and energy. For a force $\vec{\boldsymbol{F}}_{ij}$ acting on a point mass i we have from Newton's second law that:

$$\vec{\boldsymbol{F}}_{ij} = m_i \vec{\boldsymbol{a}}_i \tag{7.7}$$

Derivation. *We take the dot product of both sides of this equation with displacement* $\mathrm{d}\vec{\boldsymbol{r}}$ *and integrate over the path curve* $V r_s$ *from* s_1 *to* s_2*.*

$$\int_{\vec{\boldsymbol{r}}_s(s_1)}^{\vec{\boldsymbol{r}}_s(s_2)} \left[\vec{\boldsymbol{F}}_{ij}\right] \cdot \mathrm{d}\vec{\boldsymbol{r}} = \int_{\vec{\boldsymbol{r}}_s(s_1)}^{\vec{\boldsymbol{r}}_s(s_2)} \left[m_i \vec{\boldsymbol{a}}_i\right] \cdot \mathrm{d}\vec{\boldsymbol{r}} \tag{7.8}$$

We see directly that the left side of this equation is equal to Eq. (7.2) and can be replaced by the work $W_{ij,s_1\to s_2}$. To prove the principle in Eq. (7.1) we still have to show that the right side is equal to the change in kinetic energy.

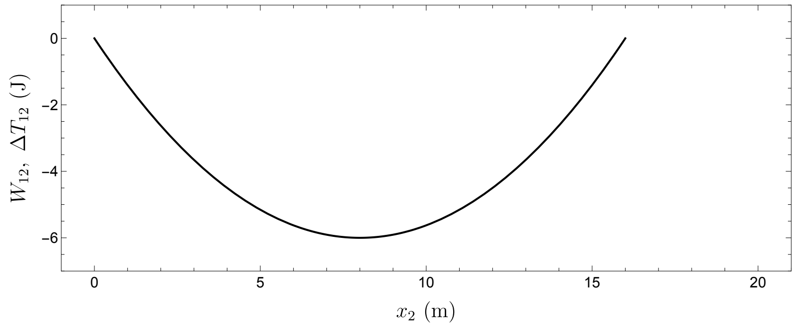


Figure 7.2: Work W_{12} done by gravitational force and change in kinetic energy ΔT_{12} of the ball whose trajectory is shown in Fig. 7.1, that travels from the origin to $x = x_2$. From the principle of work and energy $W_{12} = \Delta T_{12}$ and the curves overlap.

Derivation. *Kinetic energy*
To simplify the right side of Eq. (7.8)*, we evaluate the integral over the dot product* $\vec{\boldsymbol{a}}_i \cdot \mathrm{d}\vec{\boldsymbol{r}}$*.*

$$m_i \int_{\vec{\boldsymbol{r}}_s(s_1)}^{\vec{\boldsymbol{r}}_s(s_2)} \vec{\boldsymbol{a}}_i \cdot \mathrm{d}\vec{\boldsymbol{r}} = m_i \int_{s_1}^{s_2} \vec{\boldsymbol{a}}_i \cdot \hat{\boldsymbol{s}} \mathrm{d}s = m_i \int_{s_1}^{s_2} a_{i,t}(s) \mathrm{d}s \tag{7.9}$$

Since $a_{i,t} = \ddot{s}_i$ the rightmost integral over $a_{i,t}(s)\mathrm{d}s$ is almost identical to Eq. (5.45), and can be integrated with $a_s \mathrm{d}s = v_s \mathrm{d}v$ yielding:

$$m_i \int_{s_1}^{s_2} a_{i,t} \mathrm{d}s = m_i \int_{v_1}^{v_2} v_{s,i} \mathrm{d}v_{s,i} = \left[\frac{1}{2} m_i v_{s,i}^2\right]_{v_1}^{v_2} = \frac{1}{2} m_i v_2^2 - \frac{1}{2} m_i v_1^2 \tag{7.10}$$

The right side of this equation shows that the integral is simply equivalent to the change in the kinetic energy $T_i = \frac{1}{2} m_i v_i^2$, and proves that the integral over Newton's second law Eq. (7.8) leads to the principle of work and energy Eq. (7.1).

7.1.4 Work and energy for a system

To analyse a system of point masses, on which multiple forces F_j are working, the principle of work and energy can be extended by summing the work Eq. (7.1) over all point masses i and forces j to obtain:

$$W_{\mathrm{tot}} = \sum_i \sum_j W_{ij} = \sum_i \Delta T_i = \Delta T_{\mathrm{tot}} \tag{7.11}$$

Thus the sum of the work of all forces on all point masses equals the total increase in kinetic energy.

7.2 Conservation of energy

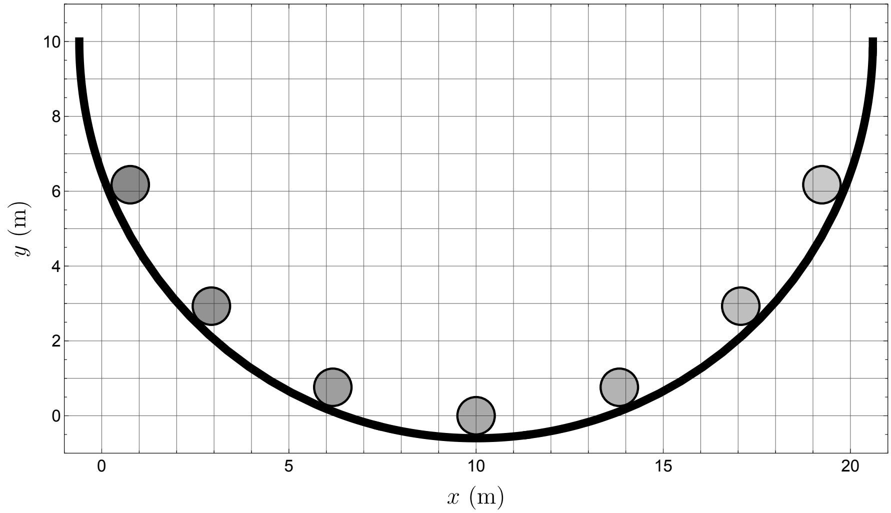


Figure 7.3: A ball in a frictionless half pipe reaches the same height on both sides.

For certain forces, so called conservative forces, conservation of energy holds. To understand what is meant by conservation of energy, consider a ball i rolling back and forth in a half pipe under the influence of gravity like in Fig. 7.3. The ball rolls up one of the ramps, stops at a maximum height, rolls downward and gains its maximum speed at the lowest point after which this sequence repeats itself. If there is no friction, the ball will continue rolling back and forth indefinitely. Apparently the system has a kind of 'memory', represented by a quantity that is always constant. Let us assume that there is a quantity called the internal energy $U_{i,\text{tot}}$ which is constant (conserved) and can be defined as the sum of the kinetic energy and a potential energy function $V_i(\vec{\boldsymbol{r}})$.

$$T_i + V_i(\vec{\boldsymbol{r}}) \equiv U_{i,\text{tot}} = \text{constant} \tag{7.12}$$

This equation is called the law of conservation of energy. We now need to derive when it holds and what are the properties of the potential energy.

If the total internal energy $U_{i,\text{tot}}$ is constant, then always $\Delta U_{i,\text{tot}} = 0$. When the point mass i moves from position $\vec{\boldsymbol{r}}_1$ to position $\vec{\boldsymbol{r}}_2$ under the influence of a force $\vec{\boldsymbol{F}}_{ij}$, this implies that the change in its kinetic and potential energy are:

$$\Delta T_{i,\vec{\boldsymbol{r}}_1 \to \vec{\boldsymbol{r}}_2} + \Delta V_{ij,\vec{\boldsymbol{r}}_1 \to \vec{\boldsymbol{r}}_2} = \Delta U_{i,\text{tot}} \equiv 0 \tag{7.13}$$

From the principle of work and energy, Eq. (7.1), we have $\Delta T_{i,\vec{r}_1\to\vec{r}_2} = W_{ij,\vec{r}_1\to\vec{r}_2}$. By combining this with Eq. (7.13) we find the equation for potential energy.

Concept. *Potential energy*
The change in the potential energy of a point mass i is the negative of the work done by a conservative force $\vec{F}_{ij,c}$ on the point mass.

$$\Delta V_{ij,\vec{r}_1\to\vec{r}_2} = -W_{ij,\vec{r}_1\to\vec{r}_2} \tag{7.14}$$

By combining Eq. (7.14) with the definition of work Eq. (7.2) we find that the potential energy obeys:

$$V_{ij}(\vec{r}_2) - V_{ij}(\vec{r}_1) = -\int_{\vec{r_1}}^{\vec{r_2}} \vec{F}_{ij,c}(\vec{r}) \cdot \mathrm{d}\vec{r} \tag{7.15}$$

However, not all forces obey Eq. (7.15). If the point mass moves along a closed path, we have $\vec{r}_2 = \vec{r}_1$. If we substitute this in Eq. (7.15) we find that only forces that satisfy the following equation are conservative forces:

Concept. *Conservative force*
The work done by a conservative force $\vec{F}_{ij,c}$ on a point mass along any closed path is zero.

$$W_{ij,s_1\to s_1} = \oint_{\vec{r_1}}^{\vec{r_1}} \vec{F}_{ij,c}(\vec{r}) \cdot \mathrm{d}\vec{r} = 0 \tag{7.16}$$

According to this definition, the gravitational force is conservative, since it follows from Eq. (7.6) that if $\vec{r}_2 = \vec{r}_1$, we have $y_2 = y_1$ and $W_{ij,s_1\to s_1} = 0$, irrespective of the path.

7.2.1 Force from potential energy

Interestingly, once the potential energy function $V_{ij}(\vec{r}_i)$ of a conservative force is known, it can be used to determine the force field vector by taking its gradient as follows:

$$\vec{F}_{ij,c}(\vec{r}) = -\vec{\nabla} V_{ij}(\vec{r}) \tag{7.17}$$

Derivation. *Here we will show how the conservative force vector can be obtained from the potential energy.*

By combining Eq. (7.4) and (7.14), we obtain for the change in potential energy when the point mass moves from $\vec{\boldsymbol{r}}_1$ to $\vec{\boldsymbol{r}}_2$:

$$\Delta V_{ij,\vec{\boldsymbol{r}}_1\to\vec{\boldsymbol{r}}_2} = -\int_{s_1}^{s_2} F_{ij,t}\mathrm{d}s \tag{7.18}$$

If one chooses the path curve to be parallel to the x axis, ds becomes dx and by substituting $s_1 = x$ and $s_2{=}x + \Delta x$ in Eq. (7.18), in the limit $\Delta x \to 0$ we obtain:

$$V_{ij}(x+\Delta x) - V_{ij}(x) = -F_{ij,x}\Delta x \tag{7.19}$$

$$\lim_{\Delta x\to 0}\frac{V_{ij}(x+\Delta x) - V_{ij}(x)}{\Delta x} = -F_{ij,x} \tag{7.20}$$

$$F_{ij,x} = -\frac{\partial V_{ij}(\vec{\boldsymbol{r}})}{\partial x} \tag{7.21}$$

A similar procedure as in Eq. (7.21) can be used to obtain the other components of the force vector, by taking the path curve parallel to y and z axes, such that the total force vector at position $\vec{\boldsymbol{r}}$ can be constructed:

$$\vec{\boldsymbol{F}}_{ij,c}(\vec{\boldsymbol{r}}) = F_{ij,x}\hat{\boldsymbol{\imath}} + F_{ij,y}\hat{\boldsymbol{\jmath}} + F_{ij,z}\hat{\boldsymbol{k}} \tag{7.22}$$

$$= -\frac{\partial V_{ij}}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial V_{ij}}{\partial y}\hat{\boldsymbol{\jmath}} - \frac{\partial V_{ij}}{\partial z}\hat{\boldsymbol{k}} \tag{7.23}$$

$$= -\vec{\boldsymbol{\nabla}}V_{ij}(\vec{\boldsymbol{r}}) \tag{7.24}$$

Here we used that the del or nabla operator $\vec{\boldsymbol{\nabla}}$ in Cartesian coordinates is $\vec{\boldsymbol{\nabla}} = \frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}$. $\vec{\boldsymbol{\nabla}}V$ is the gradient of the scalar potential energy function. Essentially, this gradient shows that the force on a point mass always points downward along the steepest 'downhill' potential direction, with a magnitude equal to the slope of the potential energy function along that direction.

7.2.2 Heat and energy conservation*

Interestingly, all fundamental forces of nature are conservative forces, which means that energy can never be lost and remains constant. Nevertheless, besides increasing the kinetic energy of large point-masses, forces like friction can also increase the random velocity of the atoms in a material. The energy associated with this random motion of the atoms is called heat Q, and leads to temperature increases. Unfortunately, since the motion is random, it is not possible to fully reuse heat to perform work, and can thus be considered

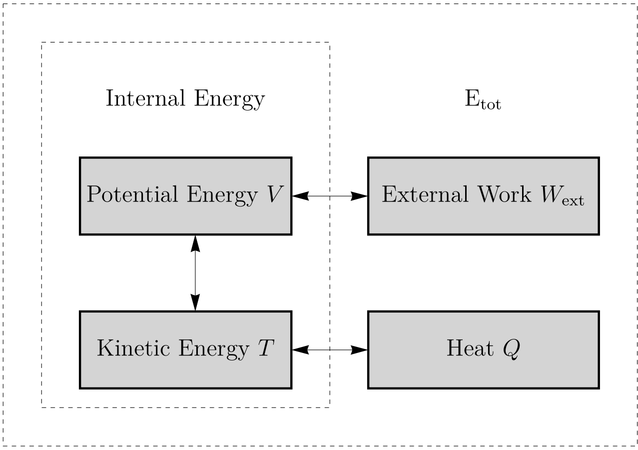


Figure 7.4: The law of conservation of internal energy ($T + V = \text{const}$) does not hold when kinetic or potential energy is converted into heat Q or used to perform work W_{ext} on an external system by non-conservative forces.

as (partly) lost energy. Forces that generate heat are therefore also called non-conservative forces or dissipative forces.

For example when a block slides off a ramp while friction forces act on it, the temperature of the block and the ramp increase due to the higher average velocity of the atoms in the materials. Non-conservative forces like friction usually generate heat ($\Delta Q > 0$). To distinguish the potential and kinetic energy stored in a closed system from heat energy and external sources or sinks of energy, the sum of kinetic and potential energy $T + V = U_i$ of a point mass or system of point masses is called its *internal energy.* Internal energy can reduce via generation of heat Q, and can also reduce by net work W_{ext} performed by the internal system on an external system. If the effect of heat and work is included, the energy conservation equation becomes:

$$T + V + Q + W_{\text{ext}} = E_{\text{tot}} = \text{constant} \tag{7.25}$$

This equation, that is depicted schematically in Fig. 7.4 is also called the first law of thermodynamics, and is discussed in more detail in thermodynamics textbooks. If we only look at changes in energy, the law can be reformulated into:

$$\Delta T + \Delta V + \Delta Q + \Delta W_{\text{ext}} = 0 \tag{7.26}$$

The equation states that total energy E_{tot} is conserved and can be interchanged between kinetic and potential energy, or heat energy Q or work W_{ext} on external systems. We note that there are always dissipative, frictional forces, which will continue to convert internal energy into heat as long as the system keeps moving. Therefore, unless there are external sources of work or heat ($\Delta Q + \Delta W_{\mathrm{ext}} < 0$), the system will eventually stop moving once the internal energy is all converted into heat. As a consequence it is impossible to create so-called *perpetuum mobiles*: mechanisms that continue moving forever without energy supply.

7.2.3 Power

Concept. *Power*
The work W of a force on a point mass can be divided by the time interval over which this work is done to obtain the average power generated by the force. When the time interval tends to zero we obtain that the (instantaneous) power generated by a force is:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{7.27}$$

From Eq. (7.2) it can be seen that $\mathrm{d}W = \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}}$, from which we find that the rate at which power is generated equals the inner product of the force and velocity vectors:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \vec{\boldsymbol{F}} \cdot \frac{\mathrm{d}\vec{\boldsymbol{r}}}{\mathrm{d}t} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} \tag{7.28}$$

Inversely, the total work done by a force can be obtained by integrating the power over time:

$$W = \int P\mathrm{d}t = \int \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\mathrm{d}t \tag{7.29}$$

7.2.4 Efficiency

Let's say we want to determine the efficiency of a car engine. The external input energy is the fuel consumed by the car engine which is converted into kinetic, potential and heat energy and is thus equal to $\Delta W_{\mathrm{in}} = \Delta T + \Delta V + \Delta Q$. The useful output energy is the kinetic energy and potential energy increase of the car $\Delta T + \Delta V$, that it uses to accelerate or to drive uphill. The non-useful energy is the heat that is generated in the car ΔQ.

The efficiency of the car engine is then defined as:

$$\eta = \frac{\text{Useful output energy}}{\text{Total input energy}} \times 100\% = \frac{\Delta T + \Delta V}{\Delta T + \Delta V + \Delta Q} \times 100\% \tag{7.30}$$

By taking the time derivative of the numerator and denominator of this equation one obtains the instantaneous efficiency:

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} \times 100\% = \frac{\text{Useful output power}}{\text{Total input power}} \times 100\% \tag{7.31}$$

The equation for efficiency can depend on the type of application. In a car, heat generation Q is not useful output energy, but in a heating appliance it is.

S **Example 7.2** A block with mass m slides from a hill. It starts from rest at a height $y = h$ and reaches a final velocity v_2 at height $y = 0$. Determine the efficiency of the conversion from potential to kinetic energy of the block.

This problem is similar to the example in the previous chapter, see Fig. 7.5. In this case, the input energy is equal to the reduction in potential energy of the block $-\Delta V = mgh$. The change in kinetic energy $\Delta T = \frac{1}{2}mv_2^2$ of the block is the useful energy. Therefore, using Eq. (7.30), we obtain:

$$\eta = \frac{\Delta T}{-\Delta V} \times 100\% = \frac{\frac{1}{2}mv_2^2}{mgh} \times 100\% \tag{7.32}$$

As a challenge, using Eqs. (6.21)–(6.23), show that for a kinetic friction coefficient μ_k and slope angle α the efficiency is:

$$\eta = \left(1 - \frac{\mu_k}{\tan\alpha}\right) \times 100\% \tag{7.33}$$

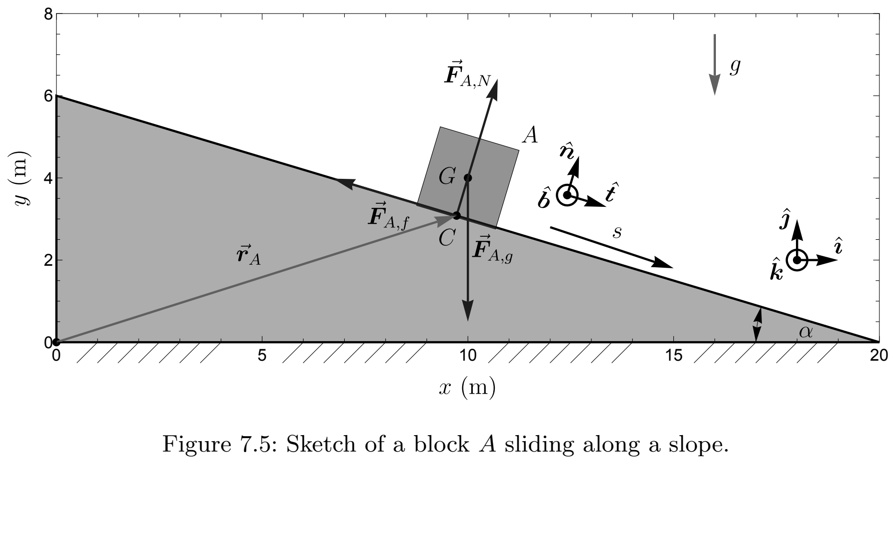


Figure 7.5: Sketch of a block A sliding along a slope.

7.2.5 Conservation of energy for a system

Let us now consider the situation when there are multiple point masses i in a system, each with a potential energy function V_i. Instead of considering them separately, it is sometimes useful to treat them together, e.g. when the forces they exert on each other are not well known. In that case we can sum all kinetic and potential energies from Eq. (7.12) at times $t = t_1$ and $t = t_2$ to obtain:

$$\sum_i T_i(t_1) + \sum_i V_i(t_1) = \sum_i T_i(t_2) + \sum_i V_i(t_2) = \sum_i E_{i,\text{tot}} \tag{7.34}$$

The total energy of the system of point masses is conserved, and is equal to $\sum_i E_{i,\text{tot}}$ at all times.

7.3 Potential energy of specific force fields

The concept of potential energy is mainly useful for conservative forces. These forces can 'store' energy and return it later. As examples we determine the potential energy of a gravitational force field and of a spring, which are both conservative forces.

7.3.1 Gravitational energy

The gravitational force $\vec{\boldsymbol{F}}_{ig} = m_i\vec{\boldsymbol{g}}$ is given by Eq. (6.27). If the gravitational force acts along the y-axis, with $\vec{\boldsymbol{g}} = -g\hat{\boldsymbol{\jmath}}$, it follows from Eq. (7.22) that:

$$\vec{\boldsymbol{F}}_{ig} = -m_i g\hat{\boldsymbol{\jmath}} = -\frac{\partial V_{ig}}{\partial y}\hat{\boldsymbol{\jmath}} \tag{7.35}$$

By integrating this equation over y we obtain the potential energy function for a mass in a constant gravitational field:

$$V_{ig}(y_i) = \int_{y_0}^{y_i} m_i g \mathrm{d}y = [m_i g y]_{y_0}^{y_i} = m_i g y_i + C \tag{7.36}$$

The integration constant $C = -m_i g y_0$ can be arbitrarily chosen by selecting a height y_0 at which the potential energy is zero. Its choice does not affect the dynamics.

7.3.2 Centre of mass and gravity

To determine the gravitational energy of objects that consist of many point masses m_i with position vectors $\vec{\boldsymbol{r}}_j$, we define the centre of mass (CoM) and its position vector $\vec{\boldsymbol{r}}_G$ for such a system.

Definition. *Centre of mass vector*
The centre-of-mass position vector $\vec{\boldsymbol{r}}_G$ of a system of point masses m_i, with a total mass m_{tot}, is the mass weighted average of all position vectors $\vec{\boldsymbol{r}}_i$, as defined by the following equations.

$$\vec{\boldsymbol{r}}_G \equiv \frac{1}{m_{\text{tot}}}\sum_i m_i\vec{\boldsymbol{r}}_i \tag{7.37}$$

$$m_{\text{tot}} = \sum_i m_i \tag{7.38}$$

$$m_{\text{tot}}\vec{\boldsymbol{r}}_G = \sum_i m_i\vec{\boldsymbol{r}}_i \tag{7.39}$$

$$m_{\text{tot}}\vec{\boldsymbol{r}}_G = \int_V \vec{\boldsymbol{r}}\rho \mathrm{d}V \tag{7.40}$$

If the object has a mass density ρ (in kg/m^3) that is distributed over a certain volume V, then the last equation gives the volume integral that determines the centre of mass. By taking the time derivatives of Eq. (7.39) we obtain useful expressions for the velocity $\vec{\boldsymbol{v}}_G$ and the acceleration vector $\vec{\boldsymbol{a}}_G$ of the CoM:

$$m_{\text{tot}}\vec{\boldsymbol{v}}_G = \sum_i m_i\vec{\boldsymbol{v}}_i \tag{7.41}$$

$$m_{\text{tot}}\vec{\boldsymbol{a}}_G = \sum_i m_i\vec{\boldsymbol{a}}_i \tag{7.42}$$

When the gravitational force $\vec{\boldsymbol{F}}_{ig} = -m_i g\hat{\boldsymbol{\jmath}}$ acts on the system, the y coordinate of every point mass is given by $y_i = \vec{\boldsymbol{r}}_i \cdot \hat{\boldsymbol{\jmath}}$, such that the potential energy of the whole system of point masses is given by:

$$V_{g,\text{tot}} = \sum_i V_{ig} = \sum_i m_i g y_i = g\sum_i m_i\vec{\boldsymbol{r}}_i \cdot \hat{\boldsymbol{\jmath}} = m_{\text{tot}}\vec{\boldsymbol{r}}_G \cdot \hat{\boldsymbol{\jmath}} = m_{\text{tot}} y_G, \tag{7.43}$$

where we used Eq. (7.39). This equation shows that the gravitational potential energy of a system of point masses is identical to the potential energy that the system would have if all point masses would be located at its centre of mass at a height y_G. The centre of mass is therefore also sometimes called the *centre of gravity*. We use the subscript G to indicate the centre of mass.

7.3.3 Spring energy

The properties of a spring that is aligned along the positive x-axis and connected at the origin can be analysed in a similar way. When the spring is displaced by a distance x_{ke} from its relaxed length, the spring force of the spring on a mass i (Eq. 6.33) is related to the potential energy:

$$\vec{\boldsymbol{F}}_{ik}(x_{ke}) = -kx_{ke}\hat{\boldsymbol{\imath}} = -\frac{\partial V_{ik}}{\partial x}\hat{\boldsymbol{\imath}} \tag{7.44}$$

By integrating this equation over x we obtain:

$$V_{ik}(x_{ke}) = \int_0^{x_{ke}} kx\mathrm{d}x = \left[\frac{1}{2}kx^2\right]_0^{x_{ke}} = \frac{1}{2}kx_{ke}^2 \tag{7.45}$$

If the length of the spring equals its rest length, $x_{ke} = (x_i - L_0) = 0$, its potential energy is chosen to be zero by starting the integral from $x = 0$.

7.3.4 Other conservative forces

Another important example of conservative forces are contact forces. Very often these forces act on objects that have zero velocity, or have a velocity perpendicular to the contact force, such that the power from the contace force is zero ($P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = 0$). In these cases the work done, and therefore the potential energy of these forces is zero. Thus, contact forces are conservative forces, since they do not change the total energy in the system.

7.4 Solving problems with work and energy

In this chapter two methods to solve dynamics problems using the concepts of work and energy have been described: the principle of work and energy and the concept of energy conservation. Both methods provide one scalar equation, Eq. (7.11) and Eq. (7.34), that can be used to determine one unknown scalar variable. For example, this unknown scalar variable can be:

- The magnitude of a final or initial velocity
- A final or initial position coordinate
- The work of a force, which can sometimes also be used to determine the magnitude of the force

The advantage of energy methods is that they allow solving a dynamics problems, without requiring one to solve differential equations. Moreover, under certain conditions (energy conservation) they provide solutions even if the path between the initial and final state is unknown, making the solutions also more generally valid.

Solving the problem then proceeds along the following steps:

1. Sketch the point masses, massless mechanisms, force vectors and constraints.
2. Choose and draw a suitable coordinate system (CS).
3. Determine the constraint equations.
4. Determine the kinetic energy function of all point masses in terms of the velocities.
5. Determine the work or potential energy function for all forces.
6. Determine the change in kinetic energy, change in potential energy, and/or the work done on masses by forces between the initial and final condition.
7. Write down the equation for the principle of work and energy, or the conservation of energy equation.
8. Determine the unknown scalar variable by solving the energy equation.

Energy methods only give the scalar values of the positions and velocity and not their directions. For instance, a mass with velocity v has the same kinetic energy as a mass with velocity $-v$, such that the direction of the velocity cannot be determined from the kinetic energy. Similarly, if the potential energy of a spring is calculated by energy methods, it cannot be determined if it is compressed or extended. Careful evaluation of the problem is needed, based on considerations other than energy, to determine those directions and to select the right solution.

7.5 Summary

In this chapter we have introduced the concepts of work, kinetic energy and potential energy and shown how they can provide a simpler route for solving certain problems in dynamics. The law of conservation of energy provides insight in the source of forces, and shows that if work is done, this drains other energy resources. We summarise the most important concepts and equations from this chapter:

- **Work and energy**
  - Work: $W_{12} = \int_{s_1}^{s_2} F_t \mathrm{d}s = \int_{\vec{\boldsymbol{r}}_1}^{\vec{\boldsymbol{r}}_2} \vec{\boldsymbol{F}} \cdot \mathrm{d}\vec{\boldsymbol{r}}$
  - Kinetic energy: $T = \frac{1}{2}mv^2$
  - Principle of work and energy: $W_{12} = \Delta T$
- **Conservation of energy**
  - A force $\vec{\boldsymbol{F}}_c(\vec{\boldsymbol{r}})$ is conservative if it does not perform work along a closed path.
  - For a system on which only conservative forces act, the internal energy is conserved: $U = T + V = \text{constant}$.
  - The change in potential energy is the negative of the work done by a conservative force: $\Delta V = -W_{12}$.
  - $\vec{\boldsymbol{F}}_c = -\vec{\boldsymbol{\nabla}} V$
  - $\Delta T + \Delta V + \Delta Q + \Delta W_{\text{ext}} = 0$
  - Power: $P = \frac{\mathrm{d}W}{\mathrm{d}t} = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$
  - Efficiency $\eta = \frac{E_{\text{in}}}{E_{\text{out}}} \times 100\% = \frac{P_{\text{in}}}{P_{\text{out}}} \times 100\%$
- **Potential energy expressions**
  - $V_g = mgy + C$
  - $V_k = \frac{1}{2}kx^2$
  - Normal contact forces are conservative.
  - Tangential (friction) contact forces are non-conservative.
  - The centre of mass and gravity: $\vec{\boldsymbol{r}}_G = \frac{1}{m}\sum_i m_i \vec{\boldsymbol{r}}_i = \int_V \vec{\boldsymbol{r}} \rho \mathrm{d}V$

8 Impulse and Momentum

In this chapter we will introduce the concepts of impulse and momentum, and show how they can be used to simplify the analysis of the kinetics of point masses. These concepts are particularly useful to analyse changes in velocity during short events, like collisions, where the position of the point masses are approximately constant.

8.1 Principle of impulse and momentum

We first define the concepts impulse and momentum, then introduce the principle of impulse and momentum and finally present the derivation of it. These concepts are also illustrated in Fig. 8.1.

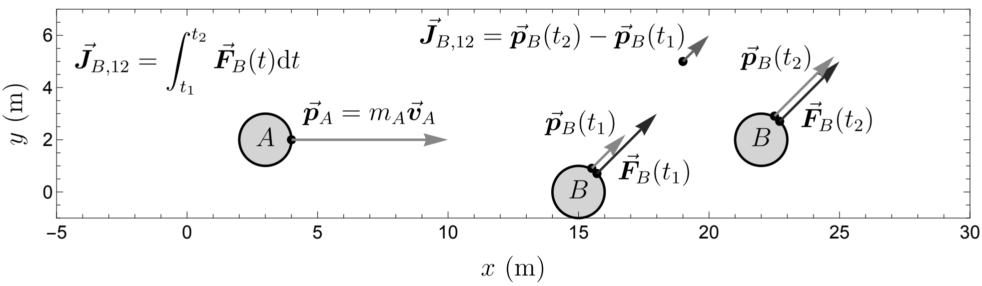


Figure 8.1: Impulse, momentum and the principle of impulse and momentum.

Concept. *Impulse*
The impulse vector $\vec{\boldsymbol{J}}_{i,12}$ *is defined as the time integral of the sum of all forces* j *acting on a mass* m_i *over a specific time interval* $t_1 - t_2$*:*

$$\vec{\boldsymbol{J}}_{i,12} \equiv \sum_j \int_{t_1}^{t_2} \vec{\boldsymbol{F}}_{ij}\mathrm{d}t \tag{8.1}$$

Impulse has N·s as unit.

Concept. *Momentum*
The momentum vector $\vec{\boldsymbol{p}}_i$ is defined as the product of a mass and its velocity vector:

$$\vec{\boldsymbol{p}}_i \equiv m_i \vec{\boldsymbol{v}}_i \tag{8.2}$$

The unit of momentum is kg·m/s.

Concept. *Principle of impulse and momentum*
The impulse acting on a mass m_i equals its change in momentum.

$$\vec{\boldsymbol{p}}_i(t_1) + \vec{\boldsymbol{J}}_{i,12} = \vec{\boldsymbol{p}}_i(t_2) \tag{8.3}$$

Since this is a vector equation, it can be projected on the three coordinate axes to obtain three scalar equations along each of the axes. The principle of impulse and momentum is mainly useful to calculate velocity changes if the impulse is known.

Derivation. *Principle of impulse and momentum*
The principle of impulse and momentum can be derived by integrating Newton's second law over time:

$$\sum_j \int_{t_1}^{t_2} \vec{\boldsymbol{F}}_{ij} \mathrm{d}t = \int_{t_1}^{t_2} m_i \vec{\boldsymbol{a}}_i \mathrm{d}t \tag{8.4}$$

The integral on the right can be evaluated to obtain:

$$\sum_j \int_{t_1}^{t_2} \vec{\boldsymbol{F}}_{ij} \mathrm{d}t = \int_{t_1}^{t_2} m \frac{\mathrm{d}\vec{\boldsymbol{v}}_i}{\mathrm{d}t} \mathrm{d}t = m\vec{\boldsymbol{v}}_i(t_2) - m\vec{\boldsymbol{v}}_i(t_1) \tag{8.5}$$

From Eq. (8.1) and Eq. (8.2) it follows that this equation is identical to Eq. (8.3).

8.2 Analysing problems with impulse

Although the principle of impulse and momentum is generally valid, it is particularly useful for analysing the dynamics under these conditions:

1. The time interval $\Delta t = t_2 - t_1$ during which a force acts on a point mass is very short.
2. The force $\vec{\boldsymbol{F}}_{ij}$ acting on the point mass i is very high compared to all other forces.

When these two conditions hold, e.g. during the short instance when a ball is hit by a baseball bat, we can apply 2 assumptions:

1. The distance the point mass travels during the short time Δt is approximately zero.
2. The impulse of all other forces can be neglected because they are much smaller than the force $\vec{\boldsymbol{F}}_{ij}$.

As a consequence of point 1, we retain information on the position of the mass without having to determine (integrate) the motion during the application of the impulse because the mass does not move. As a consequence of point 2, we can determine the impulse by integrating only force $\vec{\boldsymbol{F}}_{ij}$ during the time interval Δt and neglect all other forces.

Often segmented motion (see Sec. 5.10) is used to analyse problems with impulse and momentum. The motion in the segments before and after the impulse is analysed with the methods from the previous chapters, and only the short segment where the large force is exerted is analysed with impulse and momentum.

8.3 Impulse and momentum of a system

To extend the principle of impulse and momentum to a system of point masses, it is convenient to simplify the expression of the total momentum of such a system, by using its centre of mass.

Concept. *Momentum of a system of point masses*
The total momentum of a system of point masses $\sum_i \vec{\boldsymbol{p}}_i$ is identical to the momentum $\vec{\boldsymbol{p}}_G$ of a single point mass $m_{\text{tot}} = \sum_i m_i$ that moves along with the centre of mass of the system:

$$\sum_i \vec{\boldsymbol{p}}_i = \sum_i m_i \vec{\boldsymbol{v}}_i = \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \vec{\boldsymbol{r}}_i = \frac{\mathrm{d}}{\mathrm{d}t} m_{\text{tot}} \vec{\boldsymbol{r}}_G = m_{\text{tot}} \vec{\boldsymbol{v}}_G = \vec{\boldsymbol{p}}_G \tag{8.6}$$

Here Eq. (7.39) was used for the CoM: $\sum_i m_i \vec{\boldsymbol{r}}_i = m_{\text{tot}} \vec{\boldsymbol{r}}_G$. Let us now analyse the forces that generate impulse on the system in more detail. We can subdivide them into two groups, as shown in Fig. 8.2: the external forces $\vec{\boldsymbol{F}}_{i,\text{ext}}$ acting on the masses in the system and the internal forces $\vec{\boldsymbol{F}}_{ij,\text{int}}$ that the point masses generate on each other. For this second group of forces, we know from Newton's third law, that the force vector of mass m_j on mass m_i is equal and opposite to the force vector of m_i on mass m_j: $\vec{\boldsymbol{F}}_{ij,\text{int}} = -\vec{\boldsymbol{F}}_{ji,\text{int}}$.

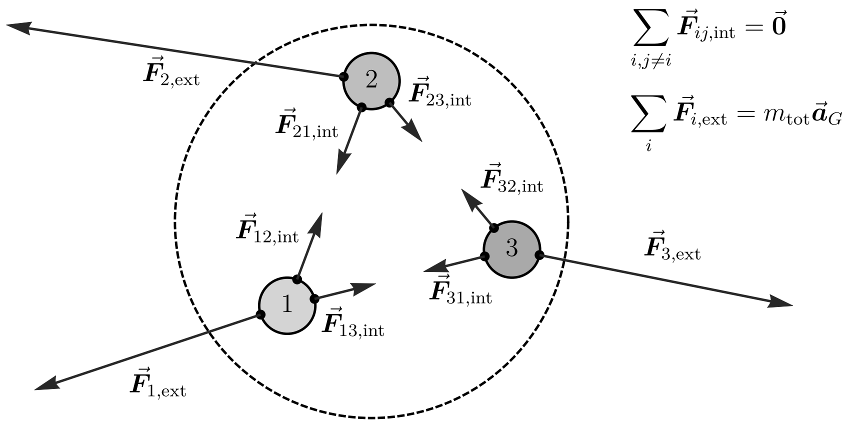


Figure 8.2: The internal and external forces on a system of point masses.

Derivation. *We are now going to derive that the sum of all internal force vectors that act between point masses in a system is zero, as can intuitively be understood from Fig. 8.2.*

To determine the total impulse on a system, we sum first over all forces j that act on a point mass to determine the total force on that point mass: $\vec{\boldsymbol{F}}_{i,\text{tot}} = \sum_{j\neq i} \vec{\boldsymbol{F}}_{ij,\text{int}} + \vec{\boldsymbol{F}}_{i,\text{ext}}$. Then we sum over all point masses i in the system: $\vec{\boldsymbol{F}}_{\text{sys,tot}} = \sum_i \vec{\boldsymbol{F}}_{i,\text{tot}}$. Finally we integrate over time and obtain the total impulse on the system:

$$\vec{\boldsymbol{J}}_{\text{sys,tot,12}} = \int_{t_1}^{t_2} \sum_i \left(\vec{\boldsymbol{F}}_{i,\text{ext}} + \sum_{j\neq i} \vec{\boldsymbol{F}}_{ij,\text{int}} \right) \mathrm{d}t \tag{8.7}$$

Now we note that the sum $\sum_i \sum_{j\neq i} \vec{\boldsymbol{F}}_{ij,\text{int}}$ includes for each term ij also a term ji with the inverted indices. Since we know from Newton's third law that $\vec{\boldsymbol{F}}_{ij,\text{int}} + \vec{\boldsymbol{F}}_{ji,\text{int}} = \vec{\boldsymbol{0}}$, the sum of internal forces in a system is always zero:

$$\sum_{i,j\neq i} \vec{\boldsymbol{F}}_{ij,\text{int}} = \vec{\boldsymbol{0}} \tag{8.8}$$

Therefore, only the impulse due to external forces contributes to the change of the total momentum. By summing Eq. (8.3) over all particles i in a system we obtain:

Concept. *Principle of impulse and momentum for a system*
The change of the total momentum of a system of point masses, during the time interval between t_1 and t_2 is identical to the sum of the impulses $\sum_i \vec{\boldsymbol{J}}_{i,\text{ext},12}$ of all acting external forces.

$$\sum_i m_i \vec{\boldsymbol{v}}_i(t_1) + \sum_i \int_{t_1}^{t_2} \vec{\boldsymbol{F}}_{i,\text{ext}} \mathrm{d}t = \sum_i m_i \vec{\boldsymbol{v}}_i(t_2) \tag{8.9}$$

$$\vec{\boldsymbol{p}}_{G,1} + \sum_i \vec{\boldsymbol{J}}_{i,\text{ext},12} = \vec{\boldsymbol{p}}_{G,2}, \tag{8.10}$$

where we used Eq. (8.1) and Eq. (8.6) to obtain an expression for the effect of an impulse on the momentum of the centre of mass of a system of point masses.

We can now derive[1] Euler's first law by taking the time derivative of Eq. (8.9) and using Eq. (8.6):

Concept. *Euler's first law*
Euler's first law states that the total mass m_{tot} times the acceleration $\vec{\boldsymbol{a}}_G$ of the CoM of a system of point masses equals the sum of external forces acting on the system:

$$\sum_i \vec{\boldsymbol{F}}_{i,\text{ext}} = m_{\text{tot}} \vec{\boldsymbol{a}}_G \tag{8.11}$$

This is an important equation, since it states that the acceleration of the CoM of a system of point masses only depends on the sum of the *external* forces on it. Interestingly, the points of action of the forces do not matter, and also internal forces can fully be neglected, which can significantly simplify analysis.

[1]We note that Euler's first law, Eq. (8.11) can also be derived by adding Newton's second law for all point masses ($\sum_{ij} \vec{\boldsymbol{F}}_{ij} = \sum_i m_i \vec{\boldsymbol{a}}_i$), and using Eqs. (8.8) and (7.42).

8.4 Law of conservation of momentum

If there are no external forces acting on the system, and no masses leave or enter the system, the system is called a *closed system.* Since $\vec{\boldsymbol{J}}_{i,\text{ext},12} = \vec{\mathbf{0}}$ in such a system, it follows from Eq. 8.10 that we have:

Concept. *Momentum conservation*
The total momentum of a closed system is conserved. This means that the total momentum is constant, equal to the momentum of the centre of mass $\vec{\boldsymbol{p}}_G$*, and does not change in time.*

$$\vec{\boldsymbol{p}}_G = \sum_i \vec{\boldsymbol{p}}_i = \text{ constant} \tag{8.12}$$

Because $\vec{\boldsymbol{p}}_G = m_{\text{tot}}\vec{\boldsymbol{v}}_G$, momentum conservation also implies that the velocity $\vec{\boldsymbol{v}}_G$ of the centre of mass of the system is constant. Since Eq. (8.12) is a vector equation it can be applied along each of the coordinate axes:

$$\sum_i p_{i,x} = \text{constant} \tag{8.13}$$

$$\sum_i p_{i,y} = \text{constant} \tag{8.14}$$

$$\sum_i p_{i,z} = \text{constant} \tag{8.15}$$

Note that if external forces only operate along one of the coordinate axes, conservation of momentum along that axis does not hold anymore, but along the other two axes momentum will still be conserved.

8.5 Collisions

An important application of the law of momentum conservation is the dynamic analysis of collisions between two objects, like the collision between two billiard balls (see Fig. 8.3). We note that methods from previous chapters often cannot be easily applied because the forces and accelerations during a collision are often not accurately known. This inaccuracy is caused by the fast dynamics and high forces during collisions, that are hard to measure. During such a short collision, other forces can usually be neglected, because they are much smaller than the very high collision forces that provide the high accelerations which change the velocity vectors in a short time. As a consequence of the absence of substantial external forces, the law of conservation of momentum Eq. (8.12) holds for the system containing both balls. We will now discuss step by step how the dynamics of the collision of two balls can be analysed using momentum conservation and the CoM-frame.

8.5.1 Plane of contact and line of impact

Let us consider two smooth spherical balls A, B with identical radius, with masses m_A, m_B and velocity vectors $\vec{\boldsymbol{v}}_A$ and $\vec{\boldsymbol{v}}_B$ that move in the xy-plane and collide, as shown in Fig. 8.3. The motion of the balls consists of 3 phases: 1. before the collision at time t_0, 2. during the collision at time t_1 and 3. after the collision at time t_2.

We can then choose our Cartesian coordinate system such that the origin O is at the point of contact. The y-axis is chosen to be tangential to the surface of the circular disks at the point where they make contact, this surface is called the *plane of contact*. The x-axis is chosen perpendicular to the plane of contact and is called the *line of impact*. The balls A and B and velocity vectors $\vec{\boldsymbol{v}}_A$, $\vec{\boldsymbol{v}}_B$ can be drawn as shown in Fig. 8.3).

The analysis of collisions becomes easier if it is performed in a CoM-frame.

Concept. *CoM-frame*
A CoM-frame is a reference frame that is chosen such that the velocity and acceleration of the CoM of the objects is zero.

The analysis process then proceeds along the following steps:

1. Determining the velocity vector $\vec{\boldsymbol{v}}_G$ of the CoM.
2. Transforming to the CoM-frame by subtracting the velocity of the centre of mass $\vec{\boldsymbol{v}}_G$ from all velocity vectors.
3. Analysing the system in the centre of mass system.
4. Transforming back to the original system by adding $\vec{\boldsymbol{v}}_G$ to all velocity vectors.

8.5.2 Transformation to the CoM-frame

Since there are no external forces acting on the balls (we neglect gravity), it is a closed system in which momentum conservation holds, such that the velocity $\vec{\boldsymbol{v}}_G$ of the centre of mass (CoM) is constant. The analysis can be significantly simplified by choosing a coordinate system x', y', z' that moves along with the CoM-frame, such that, measured in these coordinates, the velocity of the CoM is zero $\vec{\boldsymbol{v}}'_G = \vec{\boldsymbol{0}}$. A transformation to a different coordinate system like shown in Fig. 8.5 was already discussed in Eq. (6.24) and is often useful for

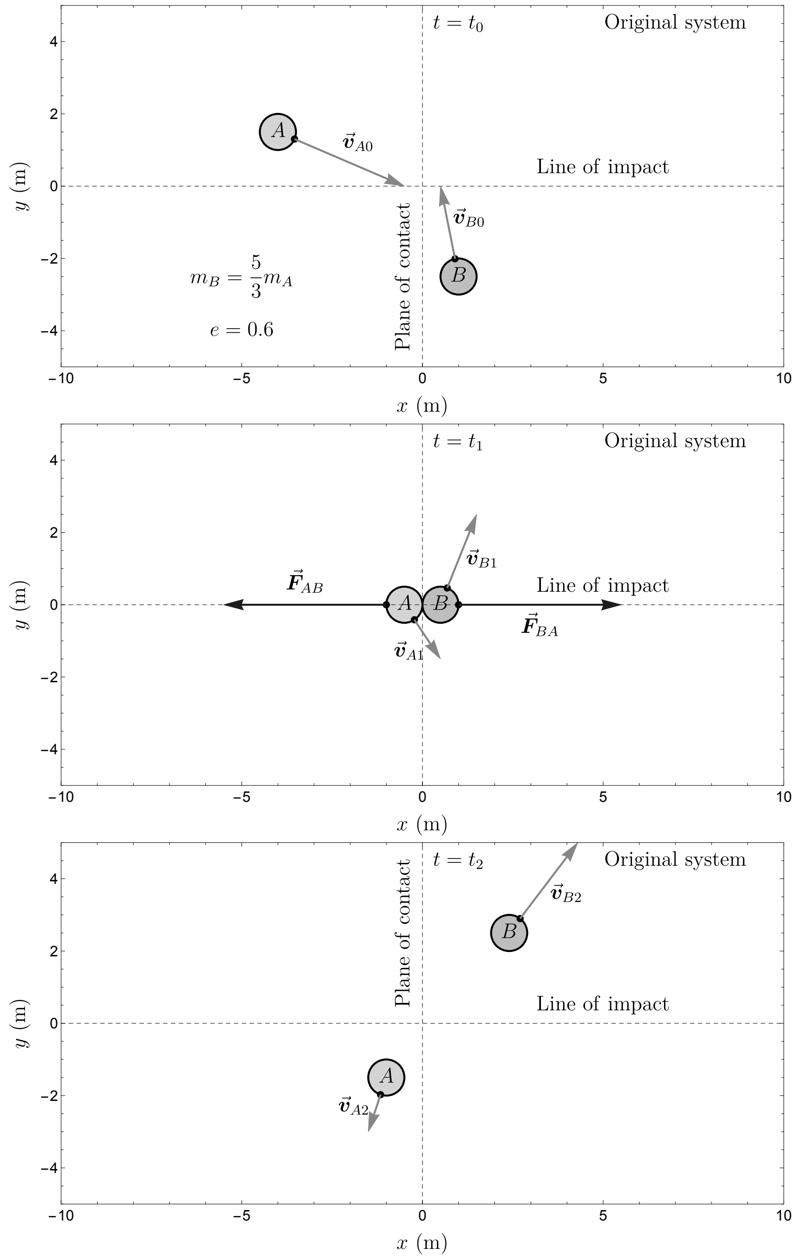


Figure 8.3: Original reference frame: motion of two balls at time t_0 before the collision, time t_1 during the collision and time t_2 after the collision. In this example we have $\vec{v}_{A0} = (3.5\hat{\imath} - 1.5\hat{\jmath})$m/s and $\vec{v}_{B0} = (-0.5\hat{\imath} + 2.5\hat{\jmath})$m/s.

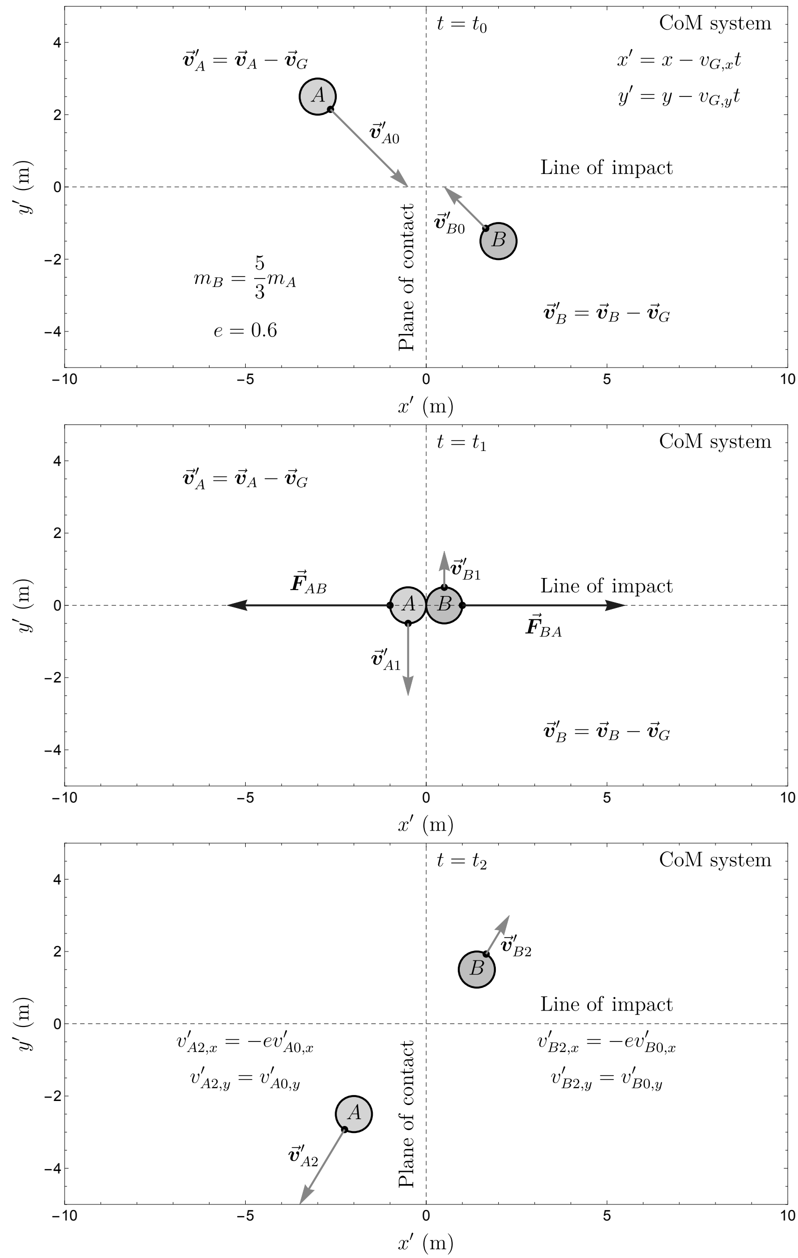


Figure 8.4: CoM-frame: now $\vec{v}'_G = \vec{0}$, so $m_A \vec{v}'_A + m_B \vec{v}'_B = \vec{0}$ at all times.

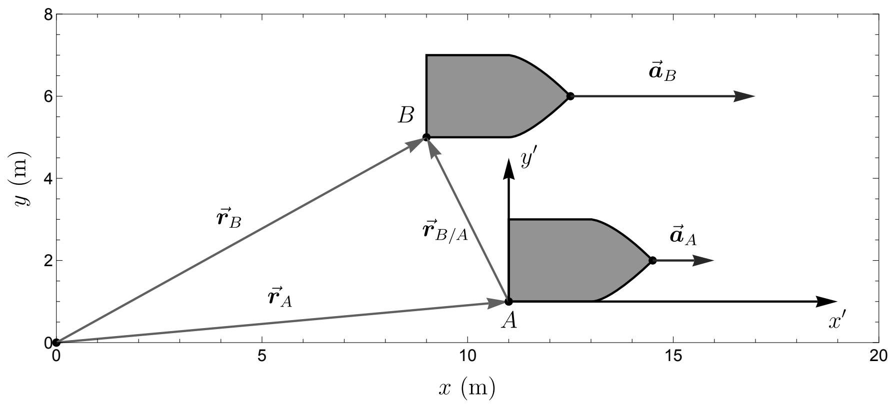


Figure 8.5: Example of a coordinate transformation to a moving reference frame x', y' with origin A. Instead of vector $\vec{r}_B$, position vector $\vec{r}_{B/A}$, with coordinates x'_B, y'_B is used to analyse the dynamics. To simplify the analysis of collisions, we choose the velocity of boat A to be equal to the constant velocity ($\vec{a}_A = \vec{0}$) of the CoM of the system of point masses ($\vec{v}_A = \vec{v}_G$) and use coordinates x' and y'.

simplifying the analysis of problems, and therefore an important technique to become familiar with (see Sec. 6.12). The position and velocity vectors in the CoM-frame are found by first determining $\vec{v}_G$ (using Eq. (7.41)) and then calculating the velocities $\vec{v}'$ in the CoM-frame by subtracting $\vec{v}_G$:

$$\vec{v}_G = \frac{m_A\vec{v}_A + m_B\vec{v}_B}{m_A + m_B} \tag{8.16}$$

$$\vec{v}'_{A0} = \vec{v}_{A0} - \vec{v}_G \tag{8.17}$$

$$\vec{v}'_{B0} = \vec{v}_{B0} - \vec{v}_G \tag{8.18}$$

$$\vec{v}'_G = \vec{v}_G - \vec{v}_G = \vec{0} \tag{8.19}$$

The result of this transformation to the CoM-frame can be seen in Fig. 8.4.

8.5.3 Analysing the collision in the CoM-frame

First we note that in the CoM-frame the velocity vectors of both balls are always parallel, pointing in opposite directions, as can be derived from $\vec{v}'_G = \vec{0}$:

$$\vec{v}'_G = \frac{m_A\vec{v}'_A + m_B\vec{v}'_B}{m_A + m_B} = \vec{0} \tag{8.20}$$

$$m_A\vec{v}'_A + m_B\vec{v}'_B = \vec{0} \tag{8.21}$$

$$\vec{v}'_B = -\frac{m_A}{m_B}\vec{v}'_A \tag{8.22}$$

Now we can analyse the collision in the CoM-frame in three steps that are illustrated in Figs. 8.4):

1. Before the impact, at time $t = t_0$ the velocity vectors $\vec{\boldsymbol{v}}'_{A0}$, $\vec{\boldsymbol{v}}'_{B0}$ point toward each other.
2. During the impact both balls exert forces $\vec{\boldsymbol{F}}_{AB} = -\vec{\boldsymbol{F}}_{BA} = F_{AB}\hat{\boldsymbol{\imath}}$ on each other along the x-axis and slightly deform at the contact point. At the time of maximal deformation $t = t_1$, the velocity x-components of both balls are zero[2] $v'_{A1,x} = v'_{B1,x} = 0$.
3. After the collision, the energy that was stored in the deformation is converted back to kinetic energy, at $t = t_2$ the velocity vectors $\vec{\boldsymbol{v}}'_{A2}$, $\vec{\boldsymbol{v}}'_{B2}$ point away from each other.

The subscripts $0, 1, 2$ are used to designate the velocities of the balls at times t_0, t_1 and t_2. To analyse the change in momentum during the impact, we apply the principle of impulse and momentum Eq. (8.9) along the y and x axes. Because the balls are smooth and frictionless, there are no forces in the y-direction tangential to the plane of contact, such that all forces act along the line of impact and are parallel to the x-axis. Since the forces and impulses along the y-axis are zero, the momentum and velocity components of both balls are conserved $\Delta p'_y = m\Delta v'_y = 0$ along that axis:

$$v'_{A2,y} = v'_{A0,y} \tag{8.23}$$

$$v'_{B2,y} = v'_{B0,y} \tag{8.24}$$

Along the x-axis it is more difficult to analyse the collision, since the contact forces along this axis are unknown, and can depend on the material properties of the balls and other conditions that affect the collision. These properties are conveniently captured by the coefficient of restitution e, that relates the velocity of the ball in the CoM at time t_2 to that at t_0:

$$v'_{A2,x} = -ev'_{A0,x} \tag{8.25}$$

$$v'_{B2,x} = -ev'_{B0,x} \tag{8.26}$$

Note that these equations follow from momentum conservation, ensuring that Eq. (8.21) still holds after the collision. The coefficient e, that can have

[2]For a collision to occur in the CoM-frame, the velocity component along the line of impact has to change sign. Therefore, there has to exist a time $t = t_1$ at which the x component of velocity is zero, and because $\vec{\boldsymbol{v}}'_G = \vec{\boldsymbol{0}}$ this occurs at the same time for both balls.

any value between 0 and 1, is closely related to the energy loss during the collision as we will discuss later. Combining Eqs. (8.23)–(8.26) we have:

$$\vec{\boldsymbol{v}}'_{A2} = -ev'_{A0,x}\hat{\boldsymbol{\imath}} + v'_{A0,y}\hat{\boldsymbol{\jmath}} \tag{8.27}$$

$$\vec{\boldsymbol{v}}'_{B2} = -ev'_{B0,x}\hat{\boldsymbol{\imath}} + v'_{B0,y}\hat{\boldsymbol{\jmath}} \tag{8.28}$$

Finally, we transform back to the original coordinate system (Fig. 8.3 at t_2) by adding the CoM velocity $\vec{\boldsymbol{v}}_G$ to all velocity vectors:

$$\vec{\boldsymbol{v}}_{A2} = \vec{\boldsymbol{v}}'_{A2} + \vec{\boldsymbol{v}}_G \tag{8.29}$$

$$\vec{\boldsymbol{v}}_{B2} = \vec{\boldsymbol{v}}'_{B2} + \vec{\boldsymbol{v}}_G \tag{8.30}$$

The whole outlined procedure can also be captured in a single set of equations:

$$\vec{\boldsymbol{v}}_{A2} = \left[-e(v_{A0,x} - v_{G,x}) + v_{G,x}\right]\hat{\boldsymbol{\imath}} + v_{A0,y}\hat{\boldsymbol{\jmath}} \tag{8.31}$$

$$\vec{\boldsymbol{v}}_{B2} = \left[-e(v_{B0,x} - v_{G,x}) + v_{G,x}\right]\hat{\boldsymbol{\imath}} + v_{B0,y}\hat{\boldsymbol{\jmath}} \tag{8.32}$$

Thus we have analysed the dynamics of collisions with *oblique impact* (impact at an angle) by only using the conservation of momentum principle, without knowing any details about the forces, except the restitution coefficient e. In the special case that the balls both move parallel to the line of impact (x-axis) they experience *central impact*, where all previous equations still hold with all velocity components along the y-axis being zero ($v_{A0,y} = v_{B0,y} = 0$). Note that since we are dealing with spherical balls, the centres of mass of both objects always coincide with the line of impact. For non-circular objects, this is not always the case, and the impact can become eccentric such that the objects can start rotating after the collision. Also if the surface of the objects is not smooth, friction forces might generate rotations under oblique impact. Impact with rotations will be discussed in chapter Ch. 12.

S **Example 8.1** Let us analyse a numerical example of the situation sketched in Fig. 8.3 and Fig. 8.4 with $e = 0.6$, $m_A = 3$ kg and $m_B = 5$ kg. First we determine the velocity vector $\vec{\boldsymbol{v}}_G$ of the centre of mass:

$$\vec{\boldsymbol{v}}_{A0} = [3.5\hat{\boldsymbol{\imath}} - 1.5\hat{\boldsymbol{\jmath}}] \text{ m/s} \tag{8.33}$$

$$\vec{\boldsymbol{v}}_{B0} = [-0.5\hat{\boldsymbol{\imath}} + 2.5\hat{\boldsymbol{\jmath}}] \text{ m/s} \tag{8.34}$$

$$\vec{\boldsymbol{v}}_G = \left[\frac{3 \cdot 3.5 + 5 \cdot -0.5}{3+5}\hat{\boldsymbol{\imath}} + \frac{3 \cdot -1.5 + 5 \cdot 2.5}{3+5}\hat{\boldsymbol{\jmath}}\right] \text{ m/s} \tag{8.35}$$

$$= [1\hat{\boldsymbol{\imath}} + 1\hat{\boldsymbol{\jmath}}] \text{ m/s} \tag{8.36}$$

Then we transform to the centre of mass system:

$$\begin{aligned}
\vec{\boldsymbol{v}}'_{A0} &= \vec{\boldsymbol{v}}_{A0} - \vec{\boldsymbol{v}}_G = [(3.5-1)\hat{\boldsymbol{\imath}} + (-1.5-1)\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.37)\\
&= [2.5\hat{\boldsymbol{\imath}} - 2.5\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.38)\\
\vec{\boldsymbol{v}}'_{B0} &= \vec{\boldsymbol{v}}_{B0} - \vec{\boldsymbol{v}}_G = [(-0.5-1)\hat{\boldsymbol{\imath}} + (2.5-1)\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.39)\\
&= [-1.5\hat{\boldsymbol{\imath}} + 1.5\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.40)
\end{aligned}$$

We multiply the x-components by $-e = -0.6$ to obtain the velocities in the CoM-frame after the collision:

$$\begin{aligned}
\vec{\boldsymbol{v}}'_{A2} &= [-e \cdot 2.5\hat{\boldsymbol{\imath}} - 2.5\hat{\boldsymbol{\jmath}}] \text{ m/s} = [-1.5\hat{\boldsymbol{\imath}} - 2.5\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.41)\\
\vec{\boldsymbol{v}}'_{B2} &= [-e \cdot -1.5\hat{\boldsymbol{\imath}} + 1.5\hat{\boldsymbol{\jmath}}] \text{ m/s} = [0.9\hat{\boldsymbol{\imath}} + 1.5\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.42)
\end{aligned}$$

Finally we add again the centre of mass velocity vector to obtain the velocities in the original system after the impact:

$$\begin{aligned}
\vec{\boldsymbol{v}}_{A2} &= \vec{\boldsymbol{v}}'_{A2} + \vec{\boldsymbol{v}}_G = [(-1.5+1)\hat{\boldsymbol{\imath}} + (-2.5+1)\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.43)\\
&= [-0.5\hat{\boldsymbol{\imath}} - 1.5\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.44)\\
\vec{\boldsymbol{v}}_{B2} &= \vec{\boldsymbol{v}}'_{B2} + \vec{\boldsymbol{v}}_G = [(0.9+1)\hat{\boldsymbol{\imath}} + (1.5+1)\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.45)\\
&= [1.9\hat{\boldsymbol{\imath}} + 2.5\hat{\boldsymbol{\jmath}}] \text{ m/s} &(8.46)
\end{aligned}$$

As an excercise, you can check if there is momentum conservation during this collision.

8.5.4 Coefficient of restitution

The coefficient of restitution e, that was introduced to capture the effect of the forces during impact, can be used as a measure for the kinetic energy that is lost during the collision in the CoM-frame. To analyse this energy loss, we use that the kinetic energy of ball A is given by:

$$T'_A = \frac{1}{2} m_A ({v'_{A,x}}^2 + {v'_{A,y}}^2) \tag{8.47}$$

If we only consider the kinetic energy $T'_{A,x}$ contributed by the x-component of the velocity vector, we find, using Eq. (8.25) that the kinetic energy after the collision is:

$$T'_{A2,x} = \frac{1}{2} m_A {v'_{A2,x}}^2 = \frac{1}{2} m_A e^2 {v'_{A0,x}}^2 = e^2 T'_{A0,x} \tag{8.48}$$

And the same equation holds for ball B. So, the x-axis kinetic energy $T'_{A,x}$

reduces by a factor e^2 during the collision, while the y-axis contribution remains constant $T'_{A2,y} = T'_{A0,y}$. The largest energy reduction occurs for $e = 0$, which is called a plastic or *perfectly inelastic collision*. Kinetic energy is conserved ($T'_{A2,x} = T'_{A0,x}$) for $e = 1$, which is called an *elastic collision*. The restitution coefficient can have any value between 0 and 1 ($0 \leq e \leq 1$). Thus the coefficient of restitution provides information on energy conservation during collisions.

8.6 Collision against a wall

A special case is if a ball A collides against a wall. In that case there is normally no momentum conservation, because the wall is connected to the ground, which can exert external forces on the system. Let us consider a reference frame in which the wall has zero velocity $\vec{\boldsymbol{v}}_W = \vec{\mathbf{0}}$. Because the mass of the wall m_W is very large compared to the mass of the ball ($m_W \gg m_A$) and $\vec{\boldsymbol{v}}_W = \vec{\mathbf{0}}$ the chosen reference frame is a CoM-frame as can be shown as follows:

$$\vec{\boldsymbol{v}}_G = \frac{m_A \vec{\boldsymbol{v}}_A + m_W \vec{\boldsymbol{v}}_W}{m_A + m_W} \approx \frac{m_A \vec{\boldsymbol{v}}_A}{m_W} \approx \vec{\mathbf{0}} \tag{8.49}$$

This shows that ball A is in a CoM-frame and therefore collides against the wall just like in Figs. 8.4. From Eq. (8.27), we find that the final velocity of ball A with initial velocity $\vec{\boldsymbol{v}}_{A0}$, after colliding with a wall parallel to the y-axis is:

$$\vec{\boldsymbol{v}}_{A2} = -e v_{A0,x} \hat{\boldsymbol{\imath}} + v_{A0,y} \hat{\boldsymbol{\jmath}} \tag{8.50}$$

8.7 Summary

In this chapter the concepts of impulse and momentum have been introduced. An impulse is the time integral of a force vector and is related to the momentum change of a point mass during a certain time interval. The change in momentum equals the impulse generated by external forces, this also holds for systems of many point masses. If the impulse, or sum of external forces is zero, the momentum of the system does not change and the law of momentum conservation applies. These principles can be applied to determine velocity changes e.g. during collisions. We summarise the most important concepts and equations from this chapter:

- **Impulse and momentum**
  - Impulse vector: $\vec{\boldsymbol{J}}_{12} = \sum_j \int_{t_1}^{t_2} \vec{\boldsymbol{F}}_{ij} \mathrm{d}t$
  - Momentum vector: $\vec{\boldsymbol{p}}_i = m_i \vec{\boldsymbol{v}}_i$

  - Principle of impulse and momentum: $\vec{\boldsymbol{p}}_i(t_1) + \vec{\boldsymbol{J}}_{12} = \vec{\boldsymbol{p}}_i(t_2)$
  - Important assumptions during impulse: 1. Force is very high and 2. time duration of impulse is very short. This allows assuming: 1. point mass is not moving during impulse. 2. impulse of other forces can be neglected during impulse.
  - Use segmented motion to analyse momentum and impulse.

- **Impulse and momentum of a system of point masses**
  - Impulse of CoM: $\vec{\boldsymbol{p}}_G = m_{\text{tot}}\vec{\boldsymbol{v}}_G = \sum_i \vec{\boldsymbol{p}}_i$
  - The sum of all internal forces in a system is zero: $\sum_{i,j\neq i} \vec{\boldsymbol{F}}_{ij,\text{int}} = \vec{\boldsymbol{0}}$
  - Principle of impulse and momentum for a system:

$$\vec{\boldsymbol{p}}_G(t_1) + \sum_i \vec{\boldsymbol{J}}_{i,\text{ext},12} = \vec{\boldsymbol{p}}_G(t_2)$$

  - Euler's first law: $\sum_i \vec{\boldsymbol{F}}_{i,\text{ext}} = m_{\text{tot}}\vec{\boldsymbol{a}}_G$

- **Momentum conservation**
  - Law of conservation of momentum: If $\vec{\boldsymbol{J}}_{i,\text{ext},12} = \vec{\boldsymbol{0}}$, then $\vec{\boldsymbol{p}}_G(t_1) = \vec{\boldsymbol{p}}_G(t_2)$.
  - And: $\vec{\boldsymbol{v}}_G = \text{constant}$.

- **Collisions**
  - Plane of contact, line of impact, draw CS.
  - Method for analysing collisions: determine $\vec{\boldsymbol{v}}_G$, transform to the CoM-frame, $v'_{x2} = -ev'_{x0}$ and $v'_{y2} = v'_{y0}$, transform back to original system.
  - Coefficient of restitution e is a measure of kinetic energy loss in the CoM-frame: $e^2 = T'_{A2,x}/T'_{A0,x}$
  - Collision against a wall: $\vec{\boldsymbol{v}}_{A2} = -ev_{A0,x}\hat{\boldsymbol{\imath}} + v_{A0,y}\hat{\boldsymbol{\jmath}}$

Part III

Rigid Body Dynamics

9 Kinematics of Rigid Bodies

9.1 Rigid bodies

After having discussed dynamics of point masses, we now turn to the analysis of rigid bodies. In this textbook we focus on the planar kinematics of rigid bodies in the xy-plane. That means that the point masses in the rigid body all move in the xy-plane with $z = 0$. However, we note that much of the presented theory is also applicable in 3D and unless explicitly indicated, e.g. with a subscript $_{2D}$, the equations in this textbook are also valid in 3D.

Concept. *Rigid body*
A rigid body F is a set of point masses (atoms) m_i which have the special property that their relative position vectors are fixed in time, such that the body is undeformable.

Every point mass m_i in the rigid body can be identified by a position vector $\vec{\boldsymbol{r}}_i$. Since the body is rigid and undeformable, the distance between every two point masses in the rigid body is constant which relates their dynamics by the following *relative constraint equation* (see Sec. 5.2.4):

$$|\vec{\boldsymbol{r}}_{i/j}| = |\vec{\boldsymbol{r}}_i - \vec{\boldsymbol{r}}_j| = \text{ constant} \tag{9.1}$$

In the next section we discuss how the orientation and position of a rigid body can be specified.

9.2 Orientation and position

The first step in the kinematic analysis of a rigid body is to have a unique description of its position and orientation. In planar kinematics, we fully determine the position of a rigid body by fixing the position vectors of 2 points in the rigid body that can be freely chosen, like points A and B of the rectangle in Fig. 9.1.

If we know position vectors $\vec{\boldsymbol{r}}_A$ and $\vec{\boldsymbol{r}}_B$ the position and orientation of the rigid body is fully determined. This requires four coordinates: x_A, y_A, x_B and y_B. However, because we know the distance between the points, we can use

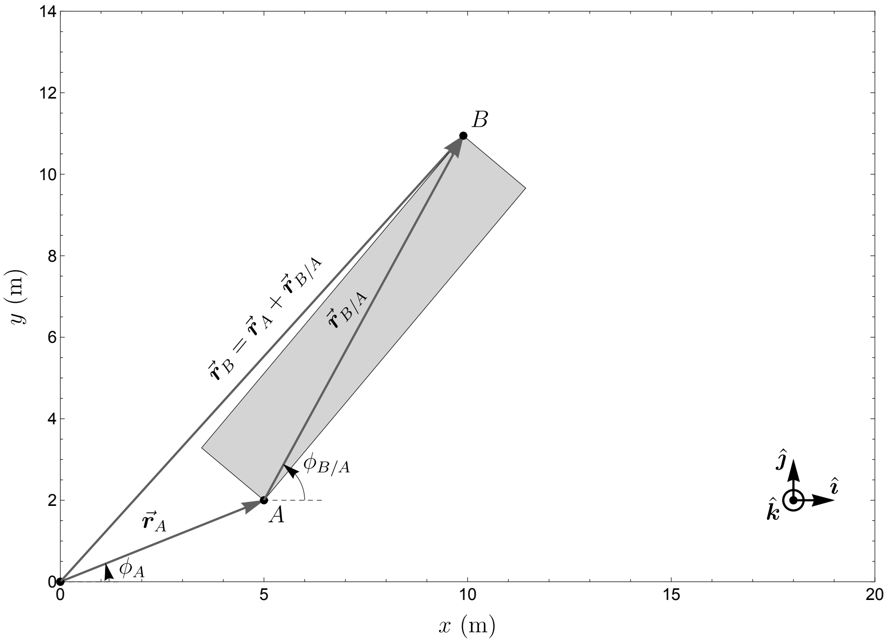


Figure 9.1: The orientation of a rigid body in the 2D xy-plane can uniquely be described by a position vector $\vec{r}_A$ and an angle $\phi_{B/A}$.

constraint equation 9.1 to reduce this to 3 coordinates, namely x_A, y_A and $\phi_{B/A}$, where $\phi_{B/A}$ is the angle the relative position vector $\vec{\boldsymbol{r}}_{B/A}$ makes with the x-axis. With x_A, y_A we can determine $\vec{\boldsymbol{r}}_A$, and with $\phi_{B/A}$ and knowledge of the distance $|\vec{\boldsymbol{r}}_{B/A}|$ we can determine $\vec{\boldsymbol{r}}_{B/A}$:

$$\vec{\boldsymbol{r}}_{A,2D} = x_A\hat{\boldsymbol{\imath}} + y_A\hat{\boldsymbol{\jmath}} \tag{9.2}$$

$$\vec{\boldsymbol{r}}_{B/A,2D} = |\vec{\boldsymbol{r}}_{B/A}|\cos\phi_{B/A}\hat{\boldsymbol{\imath}} + |\vec{\boldsymbol{r}}_{B/A}|\sin\phi_{B/A}\hat{\boldsymbol{\jmath}} \tag{9.3}$$

Now we determine $\vec{\boldsymbol{r}}_B$ from the 3 coordinates by adding these two vectors as shown in Fig. 9.1.

$$\vec{\boldsymbol{r}}_B = \vec{\boldsymbol{r}}_A + \vec{\boldsymbol{r}}_{B/A} \tag{9.4}$$

$$\vec{\boldsymbol{r}}_{B,2D} = (x_A + |\vec{\boldsymbol{r}}_{B/A}|\cos\phi_{B/A})\hat{\boldsymbol{\imath}} + (y_A + |\vec{\boldsymbol{r}}_{B/A}|\sin\phi_{B/A})\hat{\boldsymbol{\jmath}} \tag{9.5}$$

9.3 Velocities in a rigid body

Derivation. *Velocity of a point B in a rigid body*

By taking the time derivative of the position vector $\vec{\boldsymbol{r}}_B$ in Eq. (9.4), we can determine the velocity vector of point B in the rigid body as follows:

$$\vec{\boldsymbol{v}}_{B,2D} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{r}}_B = \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{r}}_A + \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{r}}_{B/A} \tag{9.6}$$

$$= \vec{\boldsymbol{v}}_A + \dot{\phi}_{B/A}|\vec{\boldsymbol{r}}_{B/A}|\left(-\sin\phi_{B/A}\hat{\boldsymbol{\imath}} + \cos\phi_{B/A}\hat{\boldsymbol{\jmath}}\right) \tag{9.7}$$

$$= \vec{\boldsymbol{v}}_A + \dot{\phi}_{B/A}|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A \tag{9.8}$$

In the last step we used Eq. (5.71) to replace the terms in brackets by $\hat{\boldsymbol{\phi}}_A$, which represents the unit vector at point B of a cylindrical coordinate system with origin A, which is why we add the subscript A to the unit vector. The velocity of point B in Eq. (9.8) can be split up in two parts: a vector $\vec{\boldsymbol{v}}_{B,\mathrm{trans}}$ related to translation and a vector $\vec{\boldsymbol{v}}_{B,\mathrm{rot}}$ related to rotation:

$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_{B,\mathrm{trans}} + \vec{\boldsymbol{v}}_{B,\mathrm{rot}} \tag{9.9}$$

$$\vec{\boldsymbol{v}}_{B,\mathrm{trans}} = \vec{\boldsymbol{v}}_A \tag{9.10}$$

$$\vec{\boldsymbol{v}}_{B,\mathrm{rot},2D} = \dot{\phi}_{B/A}|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A \tag{9.11}$$

In Figs. 9.2, 9.3 and 9.4 we show these three velocity vectors $\vec{\boldsymbol{v}}_{B,\mathrm{trans}}$, $\vec{\boldsymbol{v}}_{B,\mathrm{rot}}$ and $\vec{\boldsymbol{v}}_B$. Let us first discuss two special types of rigid body motion: pure translation and pure rotation.

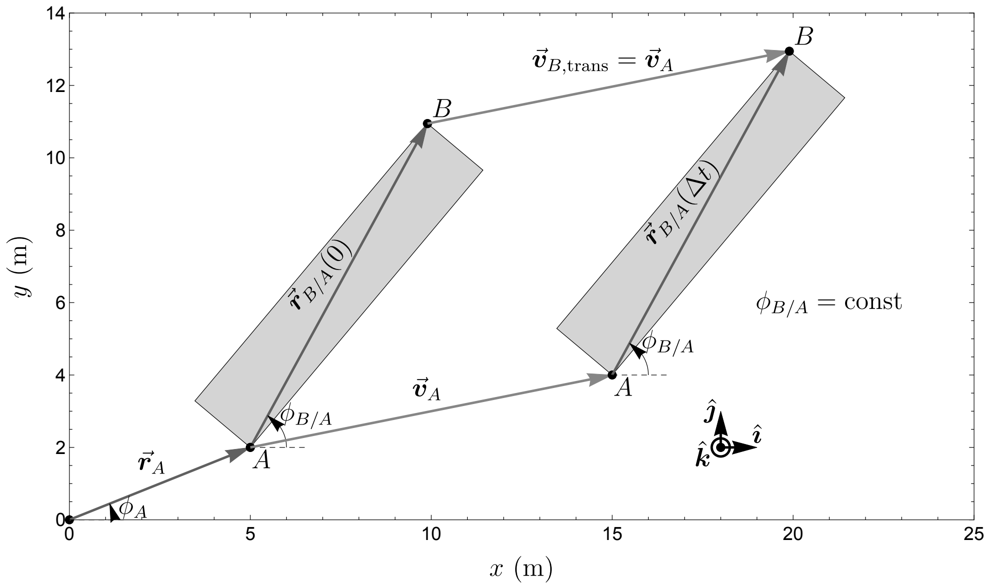


Figure 9.2: Pure translation of a rigid body: $\phi_{B/A} = \text{constant}$.

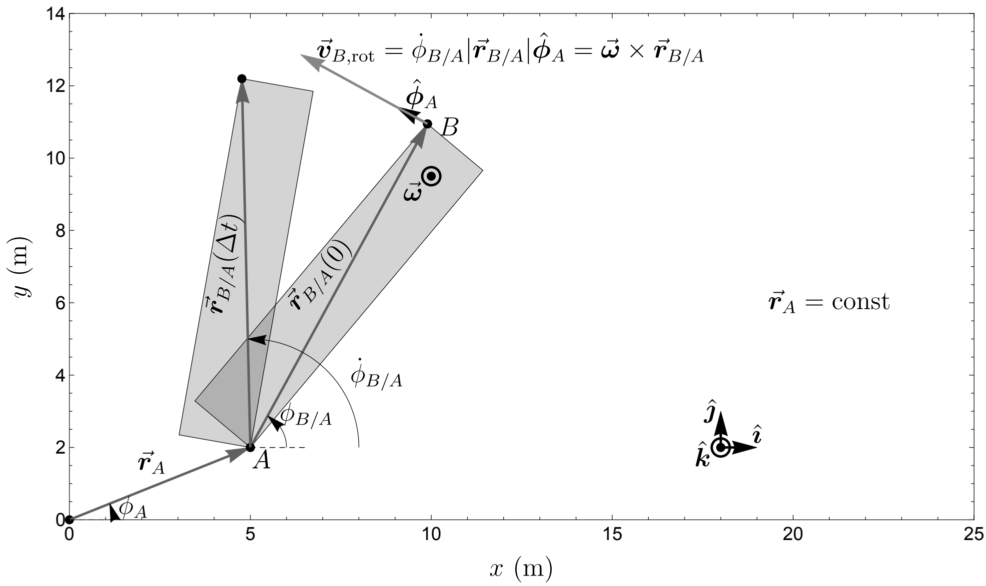


Figure 9.3: Pure rotation of a rigid body: $\vec{\boldsymbol{r}}_A = \text{constant}$.

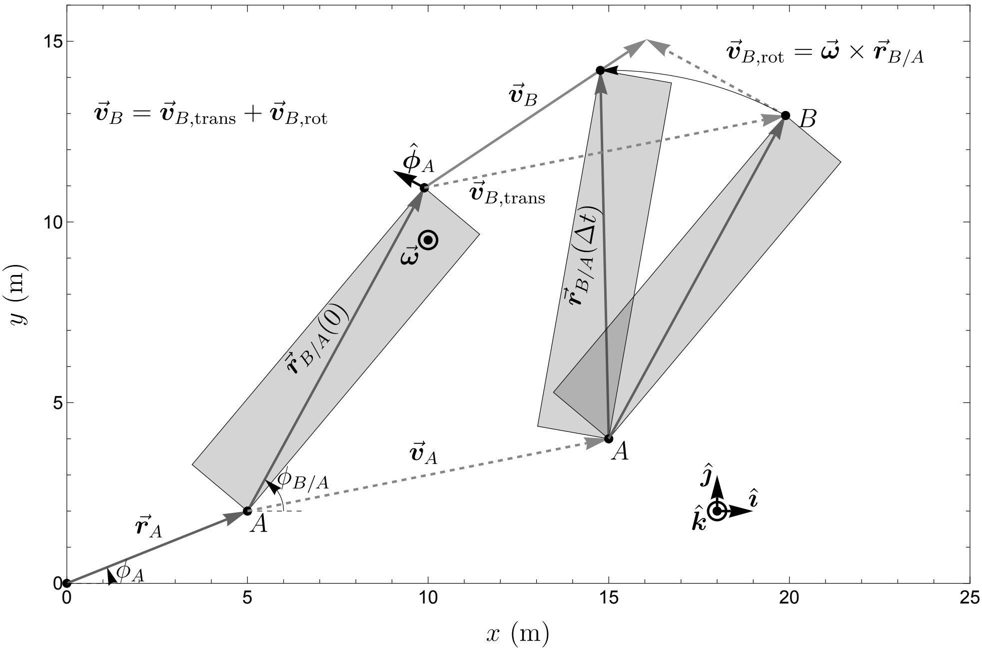


Figure 9.4: General motion of a rigid body: a combination of rotation and translation.

9.3.1 Pure translation

When the angle $\phi_{B/A}$ is kept constant, like in Fig. 9.2, all points in the rigid body move with the same velocity vector:

$$\vec{v}_B = \vec{v}_{B,\mathrm{trans}} = \vec{v}_A \tag{9.12}$$

This type of motion is called *pure translation*. The rotational component of velocity is zero, as follows from Eq. (9.11) and $\dot{\phi}_{B/A}$=0. It is important to note that pure translation can happen along any path curve $\vec{r}_A(s)$, even a circular path. The word translation thus only means that the shape of the path is identical for all points in the rigid body because $\phi_{B/A}$ is constant.

9.3.2 Pure rotation

When the position vector of a point in the rigid body is constant in time, and all other points make circular paths around it, e.g. because it rotates around an axle at that position, the motion of the rigid body is a *pure rotation*. We choose point A to be the point with constant position vector $\vec{r}_A$, as shown in Fig. 9.3. Then according to Eq. (9.11) the velocity vector of point B is given by:

$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_{B,\text{rot},2D} = \dot{\phi}_{B/A} |\vec{\boldsymbol{r}}_{B/A}| \hat{\boldsymbol{\phi}}_A \tag{9.13}$$

To facilitate the analysis of rotations and generalise it to 3D, we now define the angular velocity ω and angular velocity vector $\vec{\boldsymbol{\omega}}$.

9.3.3 Angular velocity of a rigid body

The kinematics of a rigid body is closely linked to the time derivative $\dot{\phi}_{B/A}(t)$, which is defined as its angular velocity.

Definition. *Angular velocity of a rigid body*
The time derivative $\dot{\phi}_{B/A}$ is the angular velocity ω of the rigid body:

$$\omega \equiv \dot{\phi}_{B/A} \tag{9.14}$$

The unit of angular velocity is rad/s. Interestingly, the angular velocity ω of the rigid body is independent of the choice of the points A and B on the rigid body and is therefore a general property of a rigid body as can be shown as follows.

Concept. *Independency of angular velocity*
The angular velocity ω of a rigid body is independent of the choice of points A and B on the rigid body.

Derivation. *This can be proven by drawing two relative position vectors between points $A - D$ on the rigid body, $\vec{\boldsymbol{r}}_{B/A}$ and $\vec{\boldsymbol{r}}_{D/C}$. Then the angular velocities of these straight lines $\dot{\phi}_{B/A}$ and $\dot{\phi}_{D/C}$ has to be the same, otherwise the rigid body would deform. This can be demonstrated by rotating the rigid body by a full 360 ° circle: then all points need to make a circle in the same time and therefore have the same angular velocity.*

Note that the angular velocity ω of a rigid body is different from the *orbital* angular velocity ω_o of a single point mass around an axis. Orbital angular velocity can depend on the position of the rotation axis or origin (Sec. 5.9.4). To distinguish angular velocity of a rigid body from orbital angular velocity, the angular velocity of a rigid body is therefore sometimes called its *spin* angular velocity.

9.3.4 Angular velocity vector

It can be seen in Fig. 9.3 that for pure rotation all points in the rigid body move in circular paths around point A. The velocity vector $\vec{\boldsymbol{v}}_B$ and all other points in the rigid body, lie in the plane of those circles. To describe that plane we define the angular velocity vector to be perpendicular to that plane.

Concept. *Angular velocity vector*
The angular velocity vector $\vec{\boldsymbol{\omega}}$ *of a rigid body is a vector with magnitude* $|\omega| = |\dot{\phi}_{B/A}|$ *and a direction that is perpendicular to the plane in which the rigid body rotates. Its direction can be determined using the right hand rule.*

The angular velocity vector (unit rad/s) of a rigid body that rotates in the xy plane is:

$$\vec{\boldsymbol{\omega}}_{2D} = \omega\hat{\boldsymbol{k}} = \dot{\phi}_{B/A}\hat{\boldsymbol{k}} \tag{9.15}$$

The direction of the vector can be determined using the right-hand rule by curving the fingers of your right-hand around the curved arrow in Fig. 9.3, which indicates the direction of rotational motion. Then your thumb points in the $\hat{\boldsymbol{k}}$ direction, in agreement with Eq. (9.15).

9.3.5 Determining velocities with the angular velocity vector

From Fig. 9.3 we see that the velocity vector $\vec{\boldsymbol{v}}_{B,\text{rot}}$ lies in the plane in which the rigid body moves. It is therefore perpendicular to $\vec{\boldsymbol{\omega}}$. It is also perpendicular to the vector $\vec{\boldsymbol{r}}_{B/A}$, since this vector is the radius of the circular motion. To obtain a vector that is perpendicular to two other vectors we take the cross product of these vectors:

Concept. *Rotational velocity equation*
The rotational velocity of a point B in a rigid body is given by:

$$\vec{\boldsymbol{v}}_{B,\text{rot}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \tag{9.16}$$

For the 2D case, where $\vec{\boldsymbol{\omega}}_{2D} = \dot{\phi}_{B/A}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{r}}_{B/A} = |\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\rho}}_A$, it is straightforward to check the correctness of this equation by comparison with Eq. (9.11) and by using that $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\rho}}_A = \hat{\boldsymbol{\phi}}_A$ as follows from the right-hand rule.

9.3.6 General motion

In general, as shown in Fig. 9.4, the motion of a point in a rigid body is a sum of translational and rotational motion. By combining Eqs. (9.9) and (9.16) we obtain the most general equation and important equation for the velocity in a rigid body:

$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \tag{9.17}$$

We note that this equation is valid in 3D and for any choice of the points A and B *as long as both points move along with the rigid body.* However, a smart choice of point A can simplify the analysis.

9.4 Angular acceleration of a rigid body

After having determined the velocity vector of a point in a rigid body, it is now of interest to also determine the acceleration vector of the points in the rigid body. We take the time derivative of Eq. (9.17), and use the product rule on the vector cross product to determine the acceleration vector $\vec{\boldsymbol{a}}_B$ in a rigid body.

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{v}}_B = \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{v}}_A + \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}} \times \frac{\mathrm{d}\vec{\boldsymbol{r}}_{B/A}}{\mathrm{d}t} \tag{9.18}$$

$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{B,\mathrm{rot}} \tag{9.19}$$

In this derivation we defined the angular acceleration vector $\vec{\boldsymbol{\alpha}}$ of the rigid body.

Definition. *Angular acceleration vector*
The angular acceleration vector $\vec{\boldsymbol{\alpha}}$ of a rigid body is the time derivative of its angular velocity vector.

$$\vec{\boldsymbol{\alpha}} \equiv \frac{\mathrm{d}\vec{\boldsymbol{\omega}}}{\mathrm{d}t} \tag{9.20}$$

We note that in planar kinematics this expression can be simplified:

$$\vec{\boldsymbol{\alpha}}_{2D} = \dot{\omega}\hat{\boldsymbol{k}} = \alpha\hat{\boldsymbol{k}} \tag{9.21}$$

The unit of angular acceleration is rad/s^2. We now substitute Eq. (9.16) in Eq. (9.19) and obtain the general expression for the acceleration.

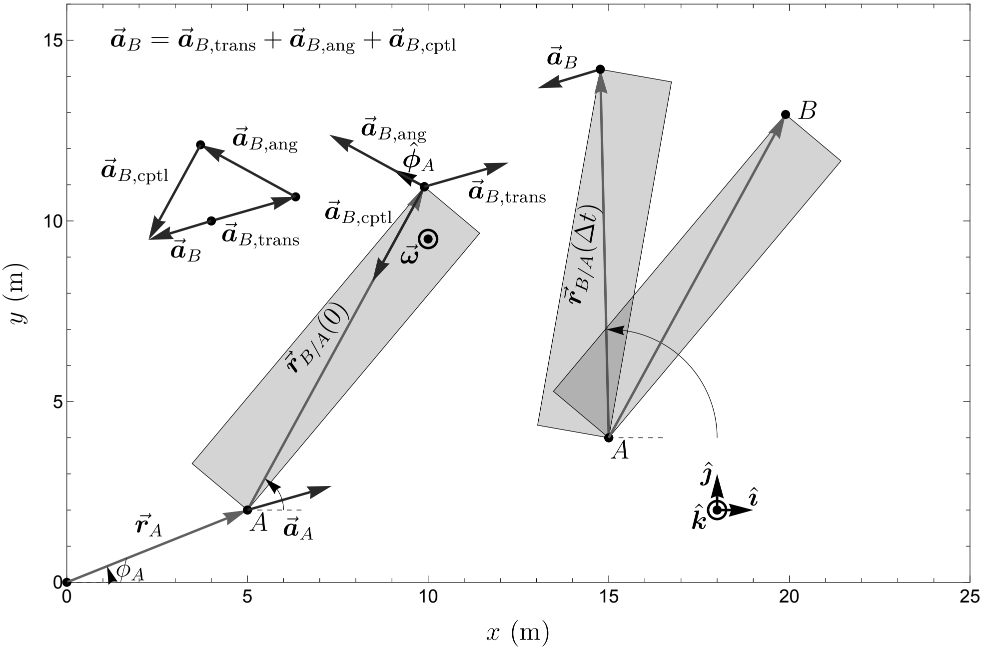


Figure 9.5: Acceleration components in a rigid body. The acceleration vector $\vec{a}_B$ is the sum of translation $\vec{a}_{B,\text{trans}} = \vec{a}_A$, angular $\vec{a}_{B,\text{ang}} = \vec{\alpha} \times \vec{r}_{B/A}$ and centripetal $\vec{a}_{B,\text{cptl}} = -\omega^2 \vec{r}_{B/A}$ acceleration vectors.

Concept. *Acceleration vector in a rigid body*
The general vector expression for the acceleration vector of a point B in a rigid body that has translational and rotational acceleration:

$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A}) \tag{9.22}$$

This equation shows that the acceleration of a point B on a rigid body consists of three contributions that are shown in Fig. 9.5.

1. The translational acceleration $\vec{\boldsymbol{a}}_{B,\mathrm{trans}} = \vec{\boldsymbol{a}}_A$ due to the acceleration of point A.
2. The angular acceleration $\vec{\boldsymbol{a}}_{B,\mathrm{ang}} = \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/A}$ due to the angular acceleration vector.
3. The centripetal acceleration $\vec{\boldsymbol{a}}_{B,\mathrm{cptl}} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A})$, due to the angular velocity vector $\vec{\boldsymbol{\omega}}$.

Acceleration in planar kinematics

In planar kinematics we can simplify the expressions for the angular acceleration somewhat. By using that $\vec{\boldsymbol{\alpha}} = \alpha\hat{\boldsymbol{k}}$ we find:

$$\vec{\boldsymbol{a}}_{B,\mathrm{ang,2D}} = \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/A} = \alpha\hat{\boldsymbol{k}} \times |\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\rho}}_A = \alpha|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A \tag{9.23}$$

The vector $\vec{\boldsymbol{\omega}}$ is always perpendicular to the xy-plane, such that $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} = \omega|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A$ and:

$$\vec{\boldsymbol{a}}_{B,\mathrm{cptl,2D}} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A}) = -\omega^2\vec{\boldsymbol{r}}_{B/A} \tag{9.24}$$

This shows that the centripetal component of acceleration always points towards point A, the centre of rotation. Note that a similar result for the centripetal acceleration term was obtained in Eq. (5.81). Combining the three terms we obtain for the planar kinematics of a rigid body the following equation:

$$\vec{\boldsymbol{a}}_{B,2D} = \vec{\boldsymbol{a}}_A + \alpha|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}} - \omega^2\vec{\boldsymbol{r}}_{B/A} \tag{9.25}$$

For completeness we repeat the most important equations for analysing the kinematics of a rigid body:

$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \tag{9.26}$$

$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}} \times \left(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A}\right) \tag{9.27}$$

$$\vec{\boldsymbol{v}}_{B,2D} = \vec{\boldsymbol{v}}_A + \omega|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A \tag{9.28}$$

$$\vec{\boldsymbol{a}}_{B,2D} = \vec{\boldsymbol{a}}_A + \alpha|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A - \omega^2\vec{\boldsymbol{r}}_{B/A} \tag{9.29}$$

9.5 Special types of motion

Let us summarise the 3D equations for the velocity and acceleration of point B in a rigid body F for the three types of motion:

1. Pure translation

$$\vec{\boldsymbol{\omega}} = \vec{\mathbf{0}} \quad (9.30)$$
$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_A \quad (9.31)$$
$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A \quad (9.32)$$

2. Pure rotation around a fixed axis in space

$$\vec{\boldsymbol{v}}_A = \vec{\mathbf{0}} \text{ and } \vec{\boldsymbol{a}}_A = \vec{\mathbf{0}} \quad (9.33)$$
$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \quad (9.34)$$
$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A}) \quad (9.35)$$

3. General motion, translation and rotation

$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \quad (9.36)$$
$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A}) \quad (9.37)$$

S **Example 9.1** As an example of the kinematic methods for rigid bodies, consider the square F in Fig. 9.6. The CoM of the square falls with a velocity $\vec{\boldsymbol{v}}_G = -1\ \hat{\boldsymbol{\jmath}}$ m/s and acceleration $\vec{\boldsymbol{a}}_G = -1\ \hat{\boldsymbol{\jmath}}$ m/s^2. At the same time the angular velocity and acceleration of the square are $\vec{\boldsymbol{\omega}} = 1\ \hat{\boldsymbol{k}}$ rad/s and $\vec{\boldsymbol{\alpha}} = 1\ \hat{\boldsymbol{k}}$ rad/s^2. The question is: *Determine the velocity and acceleration vectors of point B.*

To solve this problem, we can use Eqs. (9.36) and (9.37). Instead of point A we choose point G as reference point, because we have a lot of information on G. Then we determine the vector $\vec{\boldsymbol{r}}_{B/G} = (3.5\hat{\boldsymbol{\imath}} - 0.5\hat{\boldsymbol{\jmath}})$m from the figure. Now we use Eq. (9.36) to obtain:

$$\vec{\boldsymbol{v}}_B = [-1\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}} \times (3.5\hat{\boldsymbol{\imath}} - 0.5\hat{\boldsymbol{\jmath}})] \text{ m/s} \quad (9.38)$$
$$= (0.5\hat{\boldsymbol{\imath}} + 2.5\hat{\boldsymbol{\jmath}}) \text{ m/s} \quad (9.39)$$

And using Eq. (9.37) we obtain:

$$\vec{\boldsymbol{a}}_B = [-1\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}} \times (3.5\hat{\boldsymbol{\imath}} - 0.5\hat{\boldsymbol{\jmath}}) - 1^2(3.5\hat{\boldsymbol{\imath}} - 0.5\hat{\boldsymbol{\jmath}})] \text{ m/s}^2 \quad (9.40)$$
$$= [(0.5\hat{\boldsymbol{\imath}} + 2.5\hat{\boldsymbol{\jmath}}) - 1(3.5\hat{\boldsymbol{\imath}} - 0.5\hat{\boldsymbol{\jmath}})] \text{ m/s}^2 \quad (9.41)$$
$$= (-3\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}) \text{ m/s}^2 \quad (9.42)$$

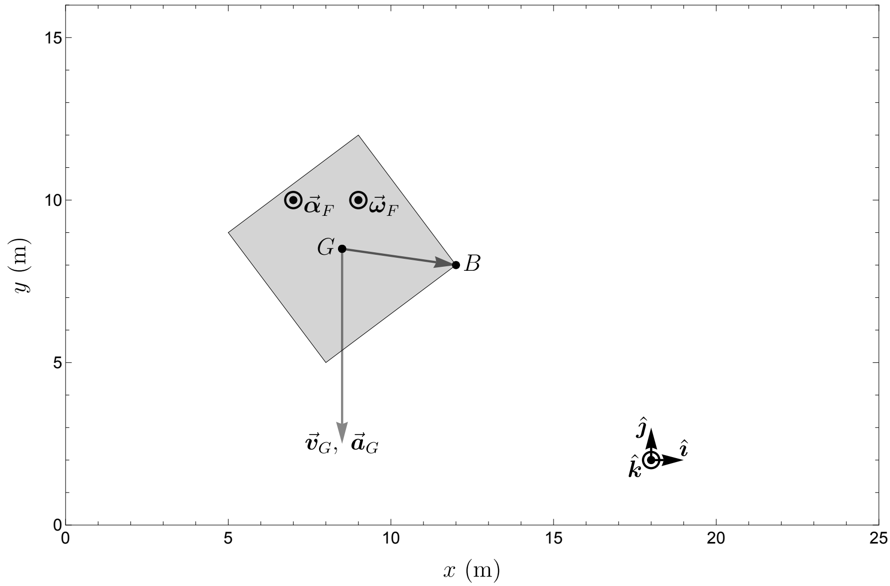


Figure 9.6: Example 9.1: a rotating square F falls downward. Determine the velocity and acceleration vector of point B.

9.5.1 Instantaneous centre of rotation

Concept. *Instantaneous centre of rotation (IC)*
In 2D kinematics it can be shown that at every instance there is always a single point in space, the instantaneous centre of rotation IC, *around which all points in the rigid body move in a circular path (pure rotation). The velocity of the rigid body at* $\vec{\boldsymbol{r}}_I C$ *is zero.*

Using the position vector $\vec{\boldsymbol{r}}_{IC}$, the velocity of all points i in the rigid body can be determined using the equation for pure rotation:

$$\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{i/IC} \tag{9.43}$$

Let us illustrate the use of the IC with an example.

S **Example 9.2** In Fig. 9.7 a wheel F with radius R rolls at an angular velocity $\vec{\boldsymbol{\omega}}$ without slip over a horizontal surface. *Find the expressions for the velocity vectors of its CoM G and point B.*

This problem can be solved by realising that the point at which the wheel touches the ground has zero velocity and is therefore the instantaneous centre of rotation IC. Now Eq. (9.43) is used to determine the velocity vector of point G and B. We have $\vec{\boldsymbol{r}}_{G/IC} = R\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{r}}_{B/IC} = R(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$ and $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. Then we have:

$$\begin{aligned}
\vec{\boldsymbol{v}}_G &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{G/IC} && (9.44)\\
&= \omega R(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}) && (9.45)\\
&= -\omega R\hat{\boldsymbol{\imath}} && (9.46)\\
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/IC} && (9.47)\\
&= \omega R(\hat{\boldsymbol{k}} \times [\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}]) && (9.48)\\
&= \omega R(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\imath}}) && (9.49)
\end{aligned}$$

So we see that the centre G of a wheel with a positive angular velocity ω rolls in the negative x direction.

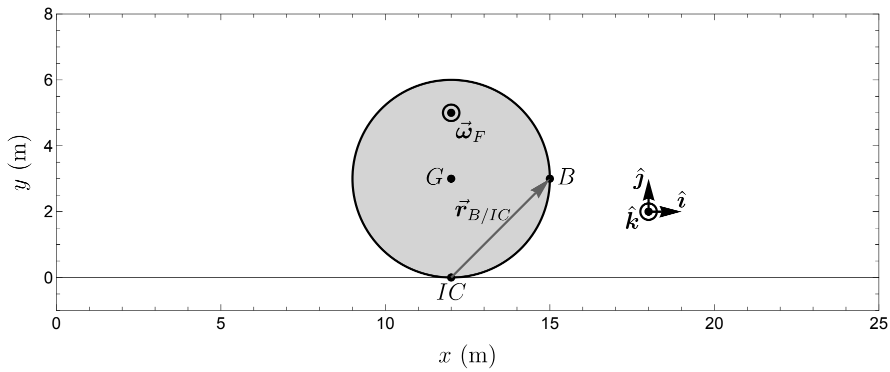


Figure 9.7: Example 9.2: a wheel rolls without slipping over a horizontal surface.

Note that the IC is not a fixed point but can move in time, like the point of contact of the wheel in the previous example. We also remark that in 3D the IC is an axis of rotation instead of a point. Let us now describe how to find the IC.

9.5.2 Finding the instantaneous centre of rotation

If a point with zero velocity is not known or present in the rigid body one can try to find the IC by utilising that all points in the rigid body move along concentric circles around point $\vec{\boldsymbol{r}}_{IC}$. This allows three methods to find the IC for planar kinematics:

1. If the angular velocity, and the velocity of a point i are known, determine the distance of that point, using the equation $|\vec{\boldsymbol{r}}_{i/IC}| = |\vec{\boldsymbol{v}}_i|/|\omega|$. Project this distance perpendicular to the vector $\vec{\boldsymbol{v}}_i$ to determine $\vec{\boldsymbol{r}}_{IC}$.

2. Use two known velocity vectors $\vec{\boldsymbol{v}}_i$ and $\vec{\boldsymbol{v}}_j$ of different points i and j on the rigid body. Determine the location $\vec{\boldsymbol{r}}_{IC}$ by drawing the lines perpendicular to these two vectors, which are parallel to the radii of the two concentric circles along which i and j move. These two lines intersect at the common centre of rotation $\vec{\boldsymbol{r}}_{IC}$.
3. If two known velocity vectors $\vec{\boldsymbol{v}}_i$ and $\vec{\boldsymbol{v}}_j$ are parallel, determine the angular velocity by using the rate at which the speed increases at larger distance from the IC: $|\omega| = (||\vec{\boldsymbol{v}}_i| - |\vec{\boldsymbol{v}}_j||)/|\vec{\boldsymbol{r}}_{i/j}|$ and continue like in point 1.

9.5.3 Choosing the reference point for rotation

As we have seen from the previous examples, multiple choices are possible for the point A that is used to analyse the kinematics of a rigid body. Ideally the expressions for the translation and rotation of the rigid body are as simple as possible.

Depending on the situation several choices for the reference point $\vec{\boldsymbol{r}}_A$ on the rigid body are possible. Often a point that is fixed in space, or does not accelerate is a good choice. Otherwise a point with zero velocity, like an IC can make the analysis simpler. Later we will see that a point where the effects of forces and moments can be easily calculated can also be good choice. In all cases, it is important to carefully consider the reference point of the rigid body, before starting to analyse its dynamics.

As illustrated in the next example, the kinematic equations discussed in this chapter can also be used to analyse the kinematics of multiple rigid bodies with relative motion that is described by constraint equations, similar to the relatively constrained kinematics for point masses discussed in Sec. 5.2.4.

S **Example 9.3** *Problem: Fig. 9.8 shows three gearwheels A, B and C, that rotate without slip. The angular velocity* $\vec{\boldsymbol{\omega}}_A = \omega_A \hat{\boldsymbol{k}}$ *and angular acceleration* $\vec{\boldsymbol{\alpha}}_A = \alpha_A \hat{\boldsymbol{k}}$ *of gearwheel A are known. Determine* ω_B, ω_C, α_B *and* α_C.

To solve this problem first determine the constraint equations. Since the gearwheels rotate without slip, there is static friction and we know from Eq. (6.28) that for the *no slip* condition, the tangential components of acceleration and velocity of the surfaces at the points where they touch are equal ($a_{P_{AB},t} = a_{P_{BA},t}$). It is important to realise that at a contact point there are points of two objects that have the same position, and one should label and distinguish those separately, like P_{AB} and P_{BA}

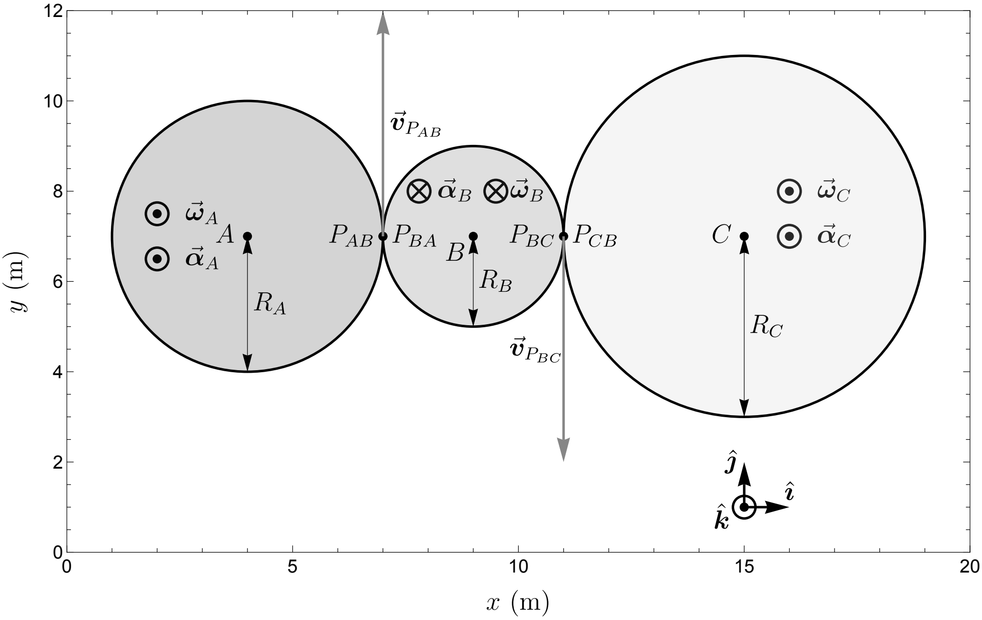


Figure 9.8: Three gearwheels A, B and C with different radii rotate without slip, such that their velocities $\vec{v}$ and tangential component of acceleration a_t are equal at the contact points. Point P_{AB} moves along with A, point P_{CB} moves along with C and points P_{BA} and P_{BC} move along with B. The teeth of the gearwheels are not shown.

in Fig. 9.8, to properly write the constraint equations:

$$\vec{\boldsymbol{v}}_{P_{AB}} = \vec{\boldsymbol{v}}_{P_{BA}} \tag{9.50}$$
$$\vec{\boldsymbol{v}}_{P_{BC}} = \vec{\boldsymbol{v}}_{P_{CB}} \tag{9.51}$$
$$a_{P_{AB},t} = a_{P_{BA},t} \tag{9.52}$$
$$a_{P_{BC},t} = a_{P_{CB},t} \tag{9.53}$$

We have 4 constraint equations and 4 unknown scalars that need to be determined, so the problem is solvable. There is pure rotation, since the gearwheels rotate around fixed axes, so we have the following kinematic equations for gearwheel A:

$$\vec{\boldsymbol{v}}_{P_{AB}} = \vec{\boldsymbol{\omega}}_A \times \vec{\boldsymbol{r}}_{P_{AB}/A} \tag{9.54}$$
$$= \omega_A \hat{\boldsymbol{k}} \times R_A \hat{\boldsymbol{\imath}} \tag{9.55}$$
$$= \omega_A R_A \hat{\boldsymbol{\jmath}} \tag{9.56}$$
$$\vec{\boldsymbol{a}}_{P_{AB}} = \vec{\boldsymbol{\alpha}}_A \times \vec{\boldsymbol{r}}_{P_{AB}/A} - \omega_A^2 \vec{\boldsymbol{r}}_{P_{AB}/A} \tag{9.57}$$
$$= \alpha_A \hat{\boldsymbol{k}} \times R_A \hat{\boldsymbol{\imath}} - \omega_A^2 R_A \hat{\boldsymbol{\imath}} \tag{9.58}$$
$$a_{P_{AB},t} = \alpha_A R_A \hat{\boldsymbol{\jmath}} \tag{9.59}$$

In the last step we separated the component of acceleration tangential to the contact surface, since that is the only direction in which the friction constrains the acceleration.

Since we know the velocity and acceleration of point P_{AB}, we can combine it with the constraint and kinematic equations to determine the angular velocity and acceleration of gearwheel B:

$$\vec{\boldsymbol{v}}_{P_{AB}} = \vec{\boldsymbol{v}}_{P_{BA}} \tag{9.60}$$
$$\omega_A R_A \hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{\omega}}_B \times \vec{\boldsymbol{r}}_{P_{BA}/B} = -\omega_B R_B \hat{\boldsymbol{\jmath}} \tag{9.61}$$
$$\omega_B = -\omega_A \frac{R_A}{R_B} \tag{9.62}$$
$$a_{P_{AB},t} = a_{P_{BA},t} \tag{9.63}$$
$$\alpha_A R_A \hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{\alpha}}_B \times \vec{\boldsymbol{r}}_{P_{BA}/B} = -\alpha_B R_B \hat{\boldsymbol{\jmath}} \tag{9.64}$$
$$\alpha_B = -\alpha_A \frac{R_A}{R_B} \tag{9.65}$$

After having obtained the angular velocity and acceleration of gearwheel B, we can follow the same procedure for wheel C.

9.6 Integration and differentiation over angles

The vector equations from the previous sections are very useful for determining the kinematics of points in the rigid body at a certain time instant. However, to predict the motion at all times, the coordinate functions $x_A(t)$, $y_A(t)$ and

$\phi_{B/A}(t)$ need to be determined. Using Eq. (9.5) we can then describe the planar time-dependent motion of a point B as follows:

$$\vec{\boldsymbol{r}}_{B,2D}(t) = \vec{\boldsymbol{r}}_A(t) + |\vec{\boldsymbol{r}}_{B/A}| \cos\phi_{B/A}(t)\hat{\boldsymbol{\imath}} + |\vec{\boldsymbol{r}}_{B/A}| \sin\phi_{B/A}(t)\hat{\boldsymbol{\jmath}} \tag{9.66}$$

The kinematics to describe the motion $\vec{\boldsymbol{r}}_A(t)$ of point A is exactly the same as for the kinematics of point masses that were described in Ch. 5, using different coordinate systems and path curves. The new aspect is that for rigid bodies we also have to describe the time dependent angle $\phi(t) \equiv \phi_{B/A}(t)$.

Luckily, for planar kinematics in 2D the required mathematics is completely the same as that for the path coordinate s_A, such that the methods we discussed in Sec. 5.7 and Sec. 5.8 can be directly applied to the angle $\phi(t)$. This thus just requires replacing s by ϕ, v by ω and a by α. For completeness we provide all the resulting equations for angles here. For time differentiation we get:

$$\phi(t) \rightarrow \tag{9.67}$$
$$\omega(t) = \dot{\phi}(t) \rightarrow \tag{9.68}$$
$$\alpha(t) = \dot{\omega}(t) = \ddot{\phi}(t) \tag{9.69}$$

For time integration we obtain:

$$\alpha(t) \rightarrow \tag{9.70}$$
$$\omega(t_2) = \omega(t_1) + \int_{t_1}^{t_2} \alpha(t)\mathrm{d}t \rightarrow \tag{9.71}$$
$$\phi(t_2) = \phi(t_1) + \int_{t_1}^{t_2} \omega(t)\mathrm{d}t \tag{9.72}$$

Similar to having functions along the path curve like $a_s(s)$ and $v_s(s)$, we can have functions for angle dependent angular velocity $\omega_\phi(\phi)$ and angular acceleration $\alpha_\phi(\phi)$. Differentiation and integration over these functions is similar to that over the path coordinate s:

$$\alpha_\phi(\phi) = \frac{\mathrm{d}\omega_\phi}{\mathrm{d}\phi}\omega_\phi(\phi) \tag{9.73}$$

$$\omega_\phi^2(\phi_2) = \omega_\phi^2(\phi_1) + 2\int_{\phi_1}^{\phi_2} \alpha_{\phi,}(\phi)\mathrm{d}\phi \tag{9.74}$$

S **Example 9.4** Gearwheel A in Fig. 9.8 has a constant acceleration α_A and $\omega_A = 0$, $\phi_A = 0$ at $t = 0$.
Determine the time dependent angle $\phi_A(t)$. It is straightforward to obtain the solution by integration:

$$\omega_A(t) = \omega_A(0) + \int_0^t \alpha_A \mathrm{d}t = \alpha_A t \tag{9.75}$$

$$\phi_A(t) = \phi_A(0) + \int_0^t \alpha_A t \mathrm{d}t = \frac{1}{2}\alpha_A t^2 \tag{9.76}$$

9.7 Motion in rotating reference frames

We will now discuss methods for analysing motion of point masses in rotating and accelerating reference frames. These methods can for instance be used to analyse mechanisms with rotating parts, and can help to predict the effect of earth's rotation on the motion of aircraft, vehicles and weather systems.

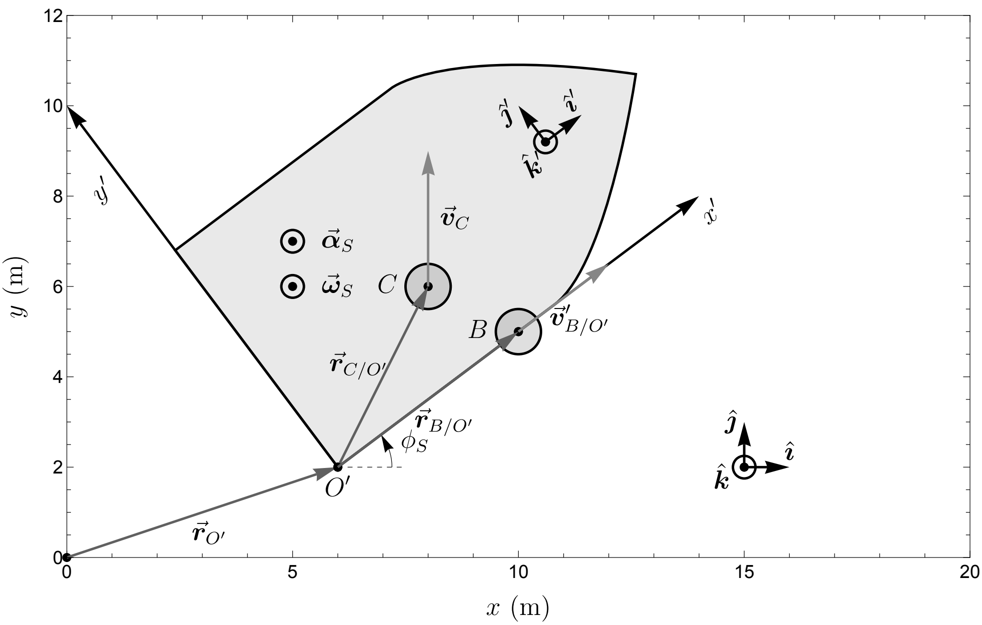


Figure 9.9: The velocity and acceleration of a ball B that rolls over the deck of a ship S can be described in the fixed coordinate system xyz, but also in the coordinate system $x'y'z'$ that is moving along with ship that is itself translating and rotating.

9.7.1 Rotating reference frame

The challenge we will deal with is illustrated in Fig. 9.9. A ball B, a point mass, rolls over the deck of a ship S and the captain of the ship measures its velocity $\vec{\boldsymbol{v}}'_{B/O'}$ and acceleration $\vec{\boldsymbol{a}}'_{B/O'}$ using the $x'y'z'$ system that moves along with the ship. An observer O on the shore also measures the velocity and acceleration of the ball and finds values $\vec{\boldsymbol{v}}_B$ and $\vec{\boldsymbol{a}}_B$, which are different from the values of the captain because the ship moves and rotates. In this section we will discuss how to relate the velocity and acceleration of the ball B as measured by the captain ($\vec{\boldsymbol{v}}'_B$, $\vec{\boldsymbol{a}}'_B$) to those measured by the observer O on the shore ($\vec{\boldsymbol{v}}_B$, $\vec{\boldsymbol{a}}_B$), if we know the motion of the ship. So we know both its translation ($\vec{\boldsymbol{v}}_{O'}$ and $\vec{\boldsymbol{a}}_{O'}$), and its angular velocity and acceleration $\vec{\boldsymbol{\omega}}_S$ and $\vec{\boldsymbol{\alpha}}_S$.

Since the mathematics to obtain the kinematic equations is rather involved, we directly present the equations here, and then derive them at the end of this chapter for interested readers.

9.7.2 Equations relating kinematics in rotating and fixed CS

With the following equations one can determine the velocity and acceleration vector of a point mass B as measured by the observer O if its velocity and acceleration as measured by the captain in the rotating and translating ship S are known:

$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{v}}_{O'} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'} \tag{9.77}$$

$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}'_{B/O'} + \vec{\boldsymbol{a}}_{O'} + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/O'} + 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{\omega}} \times \left(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'}\right) \tag{9.78}$$

$$\vec{\boldsymbol{v}}_{B,2D} = \vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{v}}_{O'} + \omega|\vec{\boldsymbol{r}}_{B/O'}|\hat{\boldsymbol{\phi}}_{O'} \tag{9.79}$$

$$\vec{\boldsymbol{a}}_{B,2D} = \vec{\boldsymbol{a}}'_{B/O'} + \vec{\boldsymbol{a}}_{O'} + \alpha|\vec{\boldsymbol{r}}_{B/O'}|\hat{\boldsymbol{\phi}}_{O'} + 2\omega(v'_{B/O',\rho}\hat{\boldsymbol{\phi}}_{O'} - v'_{B/O',\phi}\hat{\boldsymbol{\rho}}_{O'}) - \omega^2\vec{\boldsymbol{r}}_{B/O'} \tag{9.80}$$

We provide both the 3D version of the equations and the planar 2D version. Note that $\hat{\boldsymbol{\rho}}_{O'}$ and $\hat{\boldsymbol{\phi}}_{O'}$ are the unit vectors in a cylindrical coordinate system with O' as its origin. Let us now discuss the different components of these equations.

9.7.3 Object at rest in rotating reference frame

We first consider the situation where the ball is at rest on the deck of the ship at fixed position coordinate $\vec{\boldsymbol{r}}_{B/O'}$, with zero velocity $\vec{\boldsymbol{v}}'_{B/O'} = \vec{\boldsymbol{0}}$ and acceleration

$\vec{\boldsymbol{a}}'_{B/O'} = \vec{\mathbf{0}}$ in the reference frame of the ship. Substituting this in Eq. (9.77) and Eq. (9.78) we find:

$$\vec{\boldsymbol{v}}_{B,\text{rest}} = \vec{\boldsymbol{v}}_{O'} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'} \tag{9.81}$$

$$\vec{\boldsymbol{a}}_{B,\text{rest}} = \vec{\boldsymbol{a}}_{O'} + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/O'} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'}) \tag{9.82}$$

Despite the fact that the ball is at rest on the ship, these equations show us that according to the observer on the shore it can have a substantial velocity and acceleration $\vec{\boldsymbol{v}}_{B,\text{rest}}$ and $\vec{\boldsymbol{a}}_{B,\text{rest}}$. Note also that the obtained equations are identical to those for a fixed point on a rigid body, Eq. (9.26) and Eq. (9.27) as might be expected.

9.7.4 Coriolis acceleration

Now let's consider what happens when the ball B is given a velocity $\vec{\boldsymbol{v}}'_{B/O'}$ and acceleration $\vec{\boldsymbol{a}}'_{B/O'}$ on the ship deck. If we compare Eq. (9.81) and Eq. (9.77) we see that these vectors are added to vectors in rest:

$$\vec{\boldsymbol{v}}_{B} = \vec{\boldsymbol{v}}_{B,\text{rest}} + \vec{\boldsymbol{v}}'_{B/O'} \tag{9.83}$$

$$\vec{\boldsymbol{a}}_{B} = \vec{\boldsymbol{a}}_{B,\text{rest}} + \vec{\boldsymbol{a}}'_{B/O'} + \vec{\boldsymbol{a}}_{B,\text{Cor}} \tag{9.84}$$

We see that a new acceleration term is added to the acceleration equation, which is called the Coriolis acceleration.

Concept. *Coriolis acceleration*
If a point mass has a non-zero velocity $\vec{\boldsymbol{v}}'_{B/O'}$ in a reference frame that rotates with angular velocity $\vec{\boldsymbol{\omega}}$, this results in a Coriolis acceleration term that appears when the point mass is observed in a fixed reference frame:

$$\vec{\boldsymbol{a}}_{B,\text{Cor}} = 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}'_{B/O'} \tag{9.85}$$

It follows from the cross product, that the direction of the Coriolis acceleration is perpendicular to the velocity of the point mass and will therefore cause it to follow a curved circular path in the xy-plane. This rotational motion (Coriolis effect) can for instance be observed in the winds near a low pressure area as a consequence of the rotation of the earth.

9.7.5 Acceleration components in a rotating reference frame

Eq. (9.78) can be subdivided in five components, let us discuss each of them:

$$\vec{a}_B = \vec{a}'_{B/O'} + \vec{a}_{B,\text{trans}} + \vec{a}_{B,\text{ang}} + \vec{a}_{B,\text{cptl}} + \vec{a}_{B,\text{Cor}} \tag{9.86}$$

1. The acceleration $\vec{a}'_{B/O'}$ of the ball inside the rotating ship.
2. The translational acceleration $\vec{a}_{B,\text{trans}} = \vec{a}_{O'}$ due to the acceleration of the ship.
3. The angular acceleration $\vec{a}_{B,\text{ang}} = \vec{\alpha} \times \vec{r}_{B/O'}$ due to angular acceleration of the ship.
4. The centripetal acceleration $\vec{a}_{B,\text{cptl}} = \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/O'})$.
5. The Coriolis acceleration that is a consequence of the velocity $\vec{v}'_{B/O'}$ the ball has inside the rotating ship: $\vec{a}_{B,\text{Cor}} = 2\vec{\omega} \times \vec{v}'_{B/O'}$.

The equations up to now determine the acceleration $\vec{a}_B$ the observer on the shore measures. If one would like to determine the acceleration that the captain in the ship measures, this equation can be rewritten to:

$$\vec{a}'_{B/O'} = \vec{a}_B - \vec{a}_{B,\text{trans}} - \vec{a}_{B,\text{ang}} - \vec{a}_{B,\text{cptl}} - \vec{a}_{B,\text{Cor}} \tag{9.87}$$

Note that even if there are no forces acting on the ball, such that $\vec{a}_B = \vec{0}$, the captain will measure an acceleration at a rate $\vec{a}'_{B/O'}$ given by Eq. (9.87). If the captain would not know the ship is accelerating and rotating, the acceleration might be attributed to forces acting on the ball. Such forces due to measurements in a non-inertial reference frame are called pseudo-forces. Examples of such pseudo-forces are the centrifugal 'force' $-m\vec{a}_{B,\text{cptl}}$ and the Coriolis 'force' $-m\vec{a}_{B,\text{Cor}}$.

A **Example 9.5** In Fig. 9.9 a ship S is making a pure rotation in the harbour with a constant angular velocity $\vec{\omega}_S = \omega_S\hat{\boldsymbol{k}}$ and $\vec{\alpha}_S = \vec{0}$ around fixed position $\vec{r}_{O'}$. A ball B rolls over the deck of the ship with constant velocity $\vec{v'}_{B/O'} = v'_B\hat{\imath}'$ as measured by the captain in the $x'y'z'$ coordinate system that moves along with the ship. A second ball C is thrown over the ship at constant velocity $\vec{v}_C = v_C\hat{\jmath}$ as measured by an observer O on the shore. Both the captain and observer O determine the velocities and accelerations of the balls. *Determine the expressions for the velocity and and acceleration vectors that the observer and captain measure for balls B and C.*

To determine $\vec{\boldsymbol{v}}_B$ we use Eq. (9.77) with $\vec{\boldsymbol{v}}_{O'} = \vec{\mathbf{0}}$:

$$\begin{aligned} \vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{\omega}}_S \times \vec{\boldsymbol{r}}_{B/O'} && (9.88)\\ &= v'_B\hat{\boldsymbol{\imath}}' + \omega_S\hat{\boldsymbol{k}}' \times (x'_B\hat{\boldsymbol{\imath}}') && (9.89)\\ &= v'_B\hat{\boldsymbol{\imath}}' + \omega_S x'_B\hat{\boldsymbol{\jmath}}' && (9.90)\\ & && (9.91) \end{aligned}$$

Note that you can either use the unit vectors from the $x'y'z'$ or the xyz-system, as long as you make sure not to take cross or dot products between them, since the relations like Eq. (3.32) do not hold if the unit vectors are not perpendicular to each other.

To determine $\vec{\boldsymbol{a}}_B$ we use Eq. (9.78) with $\vec{\boldsymbol{a}}_{O'} = \vec{\mathbf{0}}$, $\vec{\boldsymbol{a}}_{B/O'} = \vec{\mathbf{0}}$ and $\vec{\boldsymbol{\alpha}}_S = \vec{\mathbf{0}}$:

$$\begin{aligned} \vec{\boldsymbol{a}}_B &= 2\vec{\boldsymbol{\omega}}_S \times \vec{\boldsymbol{v}}'_{B/O'} - \omega_S^2\vec{\boldsymbol{r}}_{B/O'} && (9.92)\\ &= 2\omega_S\hat{\boldsymbol{k}}' \times (v'_B\hat{\boldsymbol{\imath}}') - \omega_S^2(x'_B\hat{\boldsymbol{\imath}}') && (9.93)\\ &= 2\omega_S v'_B\hat{\boldsymbol{\jmath}}' - \omega_S^2 x'_B\hat{\boldsymbol{\imath}}' && (9.94) \end{aligned}$$

We see that the acceleration consists of a Coriolis contribution perpendicular to the velocity in the moving frame and a centripetal contribution pointing towards the moving origin O'. For the observer on the shore it is easier if we transform these vectors to the unit vectors in the xyz-system that she is using. We use the angle $\phi_S = \arctan(3/4)$ or the properties of a 3:4:5 triangle for this projection:

$$\begin{aligned} \hat{\boldsymbol{\imath}}' &= \cos(\phi_S)\hat{\boldsymbol{\imath}} + \sin(\phi_S)\hat{\boldsymbol{\jmath}} && (9.95)\\ &= 4/5\hat{\boldsymbol{\imath}} + 3/5\hat{\boldsymbol{\jmath}} && (9.96)\\ \hat{\boldsymbol{\jmath}}' &= -\sin(\phi_S)\hat{\boldsymbol{\imath}} + \cos(\phi_S)\hat{\boldsymbol{\jmath}} && (9.97)\\ &= -3/5\hat{\boldsymbol{\imath}} + 4/5\hat{\boldsymbol{\jmath}} && (9.98)\\ \hat{\boldsymbol{k}}' &= \hat{\boldsymbol{k}} && (9.99) \end{aligned}$$

However, an easier solution is to rotate the xyz-axes that the observer on the wall is using, since then we can directly set $\hat{\boldsymbol{\imath}}' = \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}' = \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}' = \hat{\boldsymbol{k}}$.

Now we are going to determine the kinematics of ball C and vectors $\vec{\boldsymbol{v}}'_{C/O'}$ and $\vec{\boldsymbol{a}}'_{C/O'}$ as observed by the captain of the ship:

$$\begin{aligned} \vec{\boldsymbol{v}}_C &= \vec{\boldsymbol{v}}'_{C/O'} + \vec{\boldsymbol{\omega}}_S \times \vec{\boldsymbol{r}}_{C/O'} && (9.100)\\ v_C\hat{\boldsymbol{\jmath}} &= \vec{\boldsymbol{v}}'_{C/O'} + \omega_S\hat{\boldsymbol{k}} \times (x_{C/O'}\hat{\boldsymbol{\imath}} + y_{C/O'}\hat{\boldsymbol{\jmath}}) && (9.101)\\ \vec{\boldsymbol{v}}'_{C/O'} &= v_C\hat{\boldsymbol{\jmath}} - \omega_S(x_{C/O'}\hat{\boldsymbol{\jmath}} - y_{C/O'}\hat{\boldsymbol{\imath}}) && (9.102)\\ &= \omega_S y_{C/O'}\hat{\boldsymbol{\imath}} + (v_C - \omega_S x_{C/O'})\hat{\boldsymbol{\jmath}} && (9.103) \end{aligned}$$

The acceleration $\vec{\boldsymbol{a}}_B$ of ball C is zero since it has a constant velocity, so we have:

$$\begin{aligned} 0 &= \vec{\boldsymbol{a}}'_{C/O'} + 2\vec{\boldsymbol{\omega}}_S \times \vec{\boldsymbol{v}}'_{C/O'} - \omega_S^2 \vec{\boldsymbol{r}}_{C/O'} && (9.104) \\ -\vec{\boldsymbol{a}}'_{C/O'} &= 2\omega_S \hat{\boldsymbol{k}} \times (\omega_S y_{C/O'} \hat{\boldsymbol{\imath}} + (v_C - \omega_S x_{C/O'})\hat{\boldsymbol{\jmath}}) - \omega_S^2 \vec{\boldsymbol{r}}_{C/O'} \\ \vec{\boldsymbol{a}}'_{C/O'} &= -2\omega_S(\omega_S y_{C/O'} \hat{\boldsymbol{\jmath}} - (v_C - \omega_S x_{C/O'})\hat{\boldsymbol{\imath}}) + \omega_S^2 (x_{C/O'} \hat{\boldsymbol{\imath}} + y_{C/O'} \hat{\boldsymbol{\jmath}}) \\ &= 2\omega_C v_C \hat{\boldsymbol{\imath}} - \omega_S^2 (x_{C/O'} \hat{\boldsymbol{\imath}} + y_{C/O'} \hat{\boldsymbol{\jmath}}) && (9.105) \\ &= 2\omega_C v_C \hat{\boldsymbol{\imath}} - \omega_S^2 \vec{\boldsymbol{r}}_{C/O'} && (9.106) \end{aligned}$$

In the second step we used Eq. (9.103). Again the acceleration consists of a Coriolis contribution perpendicular to the velocity vector in the IRF and a centripetal component that points towards the origin of the moving frame.

9.7.6 Transport theorem*

To derive Eq. (9.77) and Eq. (9.78), we will first introduce the transport theorem. The transport theorem relates the time derivative of a vector $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{f}}$ in a fixed system xyz to its time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{f}}'$ as measured in a reference frame $x'y'z'$ that rotates with angular velocity vector $\vec{\boldsymbol{\omega}}$ with respect to the fixed frame.

Concept. *Transport theorem*
The time derivative of a vector function $\vec{\boldsymbol{f}}(t)$ described in an IRF is related to the time derivative of the same vector function $\vec{\boldsymbol{f}}(t)$ as measured in a reference frame S that is rotating with angular velocity vector $\vec{\boldsymbol{\omega}}$, by the transport equation:

$$\frac{\mathrm{d}\vec{\boldsymbol{f}}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\vec{\boldsymbol{f}}}{\mathrm{d}t}\right)' + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{f}} \qquad (9.107)$$

Let us explain why the transport theorem tells us that time derivatives of vectors are different in a rotating reference frame than in a fixed reference frame. A vector represents a certain magnitude and direction in space that is independent of the coordinate system in which it is measured. However, when taking a time derivative of such a vector one actually takes the difference between two vectors measured at two *different* times t and $t + \mathrm{d}t$, e.g. for $\vec{\boldsymbol{v}}(t) = \lim_{\mathrm{d}t\to 0}(\vec{\boldsymbol{r}}(t + \mathrm{d}t) - \vec{\boldsymbol{r}}(t))/\mathrm{d}t$. If the coordinate system is rotating, the vector $\vec{\boldsymbol{r}}(t+\mathrm{d}t)$ is measured with respect to coordinate axes that have a different orientation than those for which $\vec{\boldsymbol{r}}(t)$ was measured. As a consequence, vectors that are time derivatives of other vectors, in particular $\vec{\boldsymbol{v}}(t)$ and $\vec{\boldsymbol{a}}(t)$, can be different ($\vec{\boldsymbol{v}}' \neq \vec{\boldsymbol{v}}$) if they are measured in different, relatively rotating or translating, coordinate systems and the transport theorem provides a way to relate them. In this context it is important to note that Newton's second law defines forces in terms of an acceleration vector in a non-rotating inertial reference frame (IRF). By this definition it is ensured that force vectors (and moment vectors, see Ch. 10) are independent of the coordinate system and do not need to be transformed using the transport theorem.

Derivation. *Transport theorem*
Let us derive the transport theorem for the specific case of the vector function $\vec{\boldsymbol{f}}(t)$ *that is measured both in a fixed reference frame* xyz *and in a rotating coordinate system* $x'y'z'$ *that has constant angular velocity* $\vec{\boldsymbol{\omega}}$*.*

At every time, the vector can be described as $\vec{\boldsymbol{f}} = f_x(t)\hat{\boldsymbol{\imath}}'(t) + f_j(t)\hat{\boldsymbol{\jmath}}'(t) + f_z(t)\hat{\boldsymbol{k}}'(t)$. The main point in the derivation is that for an observer that moves (and rotates) along with the rotating system $x'y'z'$ the unit vectors $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$ and $\hat{\boldsymbol{k}}'$ do not depend on time, whereas for an observer in a fixed reference frame their direction is time-dependent. The observer that moves along with the rotating coordinate system measures the time derivative of $\vec{\boldsymbol{f}}$ to be:

$$\left(\frac{\mathrm{d}\vec{\boldsymbol{f}}}{\mathrm{d}t}\right)' = \dot{f}_x\hat{\boldsymbol{\imath}}' + \dot{f}_y\hat{\boldsymbol{\jmath}}' + \dot{f}_z\hat{\boldsymbol{k}}' \tag{9.108}$$

The observer that is at rest in the fixed system xyz also measures the time derivative and finds, using the product rule:

$$\frac{\mathrm{d}\vec{\boldsymbol{f}}}{\mathrm{d}t} = (\dot{f}_x\hat{\boldsymbol{\imath}}' + \dot{f}_y\hat{\boldsymbol{\jmath}}' + \dot{f}_z\hat{\boldsymbol{k}}') + (f_x\frac{\mathrm{d}\hat{\boldsymbol{\imath}}'}{\mathrm{d}t} + f_y\frac{\mathrm{d}\hat{\boldsymbol{\jmath}}'}{\mathrm{d}t} + f_z\frac{\mathrm{d}\hat{\boldsymbol{k}}'}{\mathrm{d}t}) \tag{9.109}$$

When analysing the time derivatives of unit vectors in cylindrical coordinates we found in Eq. (5.72) relations like $\frac{\mathrm{d}\hat{\boldsymbol{\rho}}}{\mathrm{d}t} = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\rho}}$, and it can be shown that these relations hold for any unit vector in a rotating frame that is observed from a fixed reference frame like $\frac{\mathrm{d}\hat{\boldsymbol{\imath}}'}{\mathrm{d}t} = \vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}'$, such that Eq. (9.108) and (9.109) result in the transport theorem:

$$\frac{\mathrm{d}\vec{\boldsymbol{f}}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\vec{\boldsymbol{f}}}{\mathrm{d}t}\right)' + (f_x\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}' + f_y\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\jmath}}' + f_z\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}') \tag{9.110}$$

$$= \left(\frac{\mathrm{d}\vec{\boldsymbol{f}}}{\mathrm{d}t}\right)' + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{f}} \tag{9.111}$$

9.7.7 Derivation kinematics in a rotating reference frame*

Let us apply the transport theorem to the relative position vector setting $\vec{\boldsymbol{f}} = \vec{\boldsymbol{r}}_{B/O'}$ in Eq. (9.107), this yields:

$$\frac{\mathrm{d}\vec{\boldsymbol{r}}_{B/O'}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\vec{\boldsymbol{r}}_{B/O'}}{\mathrm{d}t}\right)' + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'} \tag{9.112}$$

$$\vec{\boldsymbol{v}}_{B/O'} = \vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'} \tag{9.113}$$

$$\vec{\boldsymbol{v}}_{B} = \vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{v}}_{O'} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'} \tag{9.114}$$

Where we used $\vec{\boldsymbol{v}}_{B/O} = \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_{O'}$ in the last step, thus deriving Eq. (9.77).

Derivation. *We now apply the transport theorem a second time to the velocity vector* $\vec{\boldsymbol{f}} = \vec{\boldsymbol{v}}_{B/O'}$ *to obtain the acceleration vector* $\vec{\boldsymbol{a}}_{B/O'}$*:*

$$\frac{\mathrm{d}\vec{\boldsymbol{v}}_{B/O'}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\vec{\boldsymbol{v}}_{B/O'}}{\mathrm{d}t}\right)' + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}_{B/O'} \tag{9.115}$$

$$\begin{aligned} \vec{\boldsymbol{a}}_{B/O'} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'}\right)' \\ &\quad + \vec{\boldsymbol{\omega}} \times \left(\vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'}\right) \end{aligned} \tag{9.116}$$

$$\begin{aligned} &= \vec{\boldsymbol{a}}'_{B/O'} + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/O'} + 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}'_{B/O'} \\ &\quad + \vec{\boldsymbol{\omega}} \times \left(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'}\right) \end{aligned} \tag{9.117}$$

Here we have substituted $\vec{\boldsymbol{v}}_{B/O'}$ twice using Eq. (9.113), used the product rule for differentiation, used $\vec{\boldsymbol{a}}'_{B/O'} = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{\boldsymbol{v}}'_{B/O'})'$ and $\vec{\boldsymbol{v}}'_{B/O'} = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{\boldsymbol{r}}_{B/O'})'$, and used that $\vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{\alpha}}'$ which can be shown by applying the transport theorem with $\vec{\boldsymbol{f}} = \vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\omega}} = \vec{\boldsymbol{0}}$. With Eq. (9.117) we have derived Eq. (9.78).

9.8 Summary

In this chapter we have analysed the kinematics of rotating rigid bodies. The motion of a rigid body can be considered as the combination of a translation of a certain reference point on the rigid body, and the rotation of the rigid body around an axis through that point. Velocity and acceleration vectors of the reference point can change in time, and so can the angular velocity and acceleration vectors of the rotation around the axis. In this textbook we mainly focus on the planar kinematics where the point masses move in the two-dimensional xy-plane and angular velocity and acceleration vectors point in the z-axis direction. We present the expressions for the velocities $\vec{\boldsymbol{v}}_B$ and $\vec{\boldsymbol{a}}_B$ for points in the rigid body, or point masses moving inside a rotating reference frame.

Now we are ready to analyse the dynamics of rigid bodies under the influence of forces in the next chapter, dealing with the kinetics of rigid bodies.

- **Orientation and motion of a rigid body**
  - $\vec{\boldsymbol{r}}_{B,2D}(t) = \vec{\boldsymbol{r}}_A(t) + |\vec{\boldsymbol{r}}_{B/A}| \cos\phi_{B/A}(t)\hat{\boldsymbol{\imath}} + |\vec{\boldsymbol{r}}_{B/A}| \sin\phi_{B/A}(t)\hat{\boldsymbol{\jmath}}$
- **Angular velocity and acceleration**
  - Angular velocity: $\vec{\boldsymbol{\omega}}_{2D} = \omega\hat{\boldsymbol{k}} = \dot{\phi}\hat{\boldsymbol{k}}$
  - Angular acceleration: $\vec{\boldsymbol{\alpha}}_{2D} = \alpha\hat{\boldsymbol{k}} = \ddot{\phi}\hat{\boldsymbol{k}}$
  - Angular velocity is a property of a rigid body and independent of the points used to determine the angle ϕ
  - Differentiation and integration: $\phi(t) \leftrightarrow \omega(t) \leftrightarrow \alpha(t)$
- **Kinematics rigid body**

$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \tag{9.118}$$

$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}} \times \left(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A}\right) \tag{9.119}$$

$$\vec{\boldsymbol{v}}_{B,2D} = \vec{\boldsymbol{v}}_A + \omega|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A \tag{9.120}$$

$$\vec{\boldsymbol{a}}_{B,2D} = \vec{\boldsymbol{a}}_A + \alpha|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A - \omega^2\vec{\boldsymbol{r}}_{B/A} \tag{9.121}$$

- **Special types of motion and methods**
  - Pure translation: $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{0}}$ and $\vec{\boldsymbol{v}}_A \neq \vec{\boldsymbol{0}}$.
  - Pure rotation: $\vec{\boldsymbol{v}}_A = \vec{\boldsymbol{0}}$, $\vec{\boldsymbol{a}}_A = \vec{\boldsymbol{0}}$ and $\vec{\boldsymbol{\omega}} \neq \vec{\boldsymbol{0}}$.
  - Instantaneous centre of rotation: $\vec{\boldsymbol{v}}_{IC} = \vec{\boldsymbol{0}}$.
  - 3 methods to find IC.
- **Combining kinematics with constraint equations**
  - Analyse mechanisms using multiple kinematic equations.
  - Combine these kinematic equations with constraint equations to determine the motion of mechanisms and rigid bodies. See e.g. Example 9.3.
- **Motion in a rotating reference frame x'y'z'**

$$\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{v}}_{O'} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'} \tag{9.122}$$

$$\begin{aligned}\vec{\boldsymbol{a}}_B = {} & \vec{\boldsymbol{a}}'_{B/O'} + \vec{\boldsymbol{a}}_{O'} + \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{B/O'} \\ & + 2\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{\omega}} \times \left(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/O'}\right)\end{aligned} \tag{9.123}$$

$$\vec{\boldsymbol{v}}_{B,2D} = \vec{\boldsymbol{v}}'_{B/O'} + \vec{\boldsymbol{v}}_{O'} + \omega|\vec{\boldsymbol{r}}_{B/O'}|\hat{\boldsymbol{\phi}}_{O'} \tag{9.124}$$

$$\begin{aligned}\vec{\boldsymbol{a}}_{B,2D} = {} & \vec{\boldsymbol{a}}'_{B/O'} + \vec{\boldsymbol{a}}_{O'} + \alpha|\vec{\boldsymbol{r}}_{B/O'}|\hat{\boldsymbol{\phi}}_{O'} \\ & + 2\omega(v'_{B/O',\rho}\hat{\boldsymbol{\phi}}_{O'} - v'_{B/O',\phi}\hat{\boldsymbol{\rho}}_{O'}) - \omega^2\vec{\boldsymbol{r}}_{B/O'}\end{aligned} \tag{9.125}$$

10 Kinetics of Rigid Bodies

In the previous chapter we have derived methods for analysing the kinematics of rigid bodies. Now we will use Newton's laws to evaluate how rigid bodies move in the presence of forces and derive their equations of motion. Whereas point masses can only translate, rigid bodies can also rotate, the key challenge we will therefore address is how rigid bodies rotate under the influence of forces.

10.1 Effect of a force on a rigid body

A rigid body is nothing else than a collection of many point masses that are rigidly connected and relatively constrained. So it can be analysed by the techniques we discussed earlier in Ch. 6. In this chapter we will introduce new techniques to facilitate such analysis, but before introducing these new techniques, let us give an example where we analyse the kinetics of a rigid body using the techniques from Ch. 6.

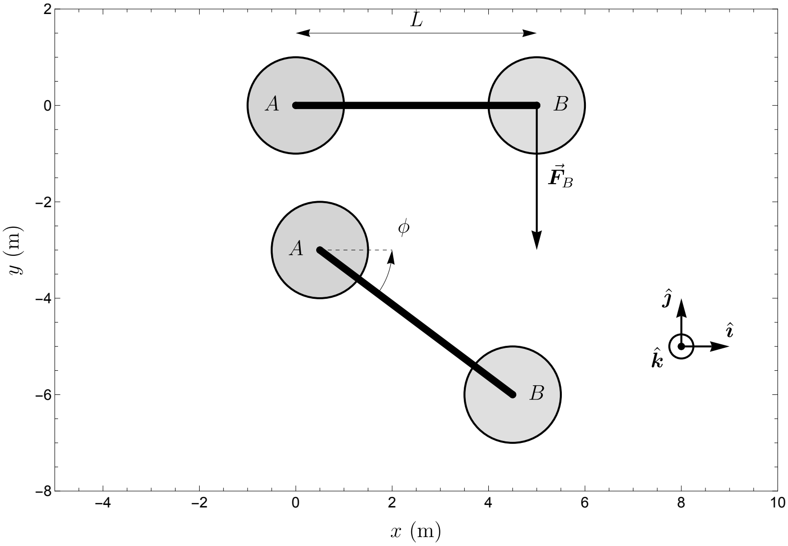


Figure 10.1: Rigid body consisting of 2 point masses connected by a rod of length L.

M **Example 10.1** Problem: Consider the rigid body C in Fig. 10.1 that consists of 2 point masses A and B connected by a rod of length L. A force $\vec{\boldsymbol{F}}_B = -F_B\hat{\boldsymbol{\jmath}}$ acts on mass B.

- *Determine the angular acceleration of C if the horizontal rigid body, with $\phi = 0$ is released from rest.*

To solve this problem we first consider the constraint equation of the rod (constant length L) and take its time derivative twice:

$$
\begin{aligned}
|\vec{\boldsymbol{r}}_{B/A}|^2 &= L^2 && (10.1)\\
x_{B/A}^2 + y_{B/A}^2 &= L^2 && (10.2)\\
2x_{B/A}\dot{x}_{B/A} + 2y_{B/A}\dot{y}_{B/A} &= 0 && (10.3)\\
\dot{x}_{B/A}^2 + x_{B/A}\ddot{x}_{B/A}^2 + \dot{y}_{B/A}^2 + y_{B/A}\ddot{y}_{B/A}^2 &= 0 && (10.4)\\
x_B(\ddot{x}_B - \ddot{x}_A)^2 &= 0 && (10.5)\\
\ddot{x}_B &= \ddot{x}_A = 0 && (10.6)
\end{aligned}
$$

In the last steps we used that for the current position and at zero velocity the constants $y_{B/A} = x_A = \dot{x}_{B/A} = \dot{y}_{B/A} = 0$ are zero. The last equation shows that if the rod is horizontal and has zero velocity, then the constraint equalises accelerations of the point masses in the x direction, but not in the y direction. Since there are no forces acting in the x-direction, we conclude that the masses don't accelerate in that direction.

Now we write down the equation of motion in the y direction for mass A and mass B:

$$
\begin{aligned}
\sum F_{A,y} &= 0 = m_A\ddot{y}_A && (10.7)\\
\sum F_{B,y} &= -F_B = m_B\ddot{y}_B && (10.8)
\end{aligned}
$$

Now the final step is to determine the angular acceleration α_C using the kinematic equations from the previous chapter. Using that $\vec{\boldsymbol{a}}_A = \vec{\boldsymbol{0}}$, $\omega_C = 0$ and $\hat{\boldsymbol{\phi}}_A = \hat{\boldsymbol{\jmath}}$ we get:

$$
\begin{aligned}
\vec{\boldsymbol{a}}_{B,2D} &= \vec{\boldsymbol{a}}_A + \alpha_C|\vec{\boldsymbol{r}}_{B/A}|\hat{\boldsymbol{\phi}}_A - \omega_C^2\vec{\boldsymbol{r}}_{B/A} && (10.9)\\
-F_B/m_B\ \hat{\boldsymbol{\jmath}} &= \alpha_C L\ \hat{\boldsymbol{\jmath}} && (10.10)\\
\alpha_C &= -F_B/(m_B L) && (10.11)
\end{aligned}
$$

Although this example demonstrates that it is possible to analyse the the effect of forces on the dynamics of rigid bodies with Newton's laws, both translation and rotation, it becomes very elaborate for rigid bodies consisting out of many point masses. In this chapter we will derive a simpler method to do this.

10.2 Kinetics of the CoM and Euler's first law

In the previous chapter we saw that only 3 coordinates are needed to determine the position and orientation of a rigid body F: x_A, y_A and ϕ_F. If we set point A at the CoM ($A = G$) and apply Euler's first law Eq. (8.11) to the rigid body, we can directly obtain the equations of motion for the coordinates of the CoM x_G and y_G:

$$\sum \vec{\boldsymbol{F}}_{\text{ext}} = m_{\text{tot}} \vec{\boldsymbol{a}}_G \quad (10.12)$$

$$\sum F_{\text{ext},x} = m_{\text{tot}} \ddot{x}_G \quad (10.13)$$

$$\sum F_{\text{ext},y} = m_{\text{tot}} \ddot{y}_G \quad (10.14)$$

Since we can use these equations (Euler's first law) to determine the trajectory $\vec{\boldsymbol{r}}_G$ of the centre of mass G of a rigid body using kinematics, we now only have to determine the angular motion $\phi(t)$ to fully analyse the motion of the rigid body and determine its position and orientation at every time using kinematic equations like Eq. (9.4). Determining the effect of forces on the time dependent rotation angles $\phi(t)$ will be the main objective of this chapter.

10.3 Rotation analysis in kinetics

Since the derivation of the key equations governing rotations in rigid body dynamics is quite elaborate, we start by introducing the equations in this section without deriving them. Then we explain how they should be used in kinetic analysis of rigid bodies and finally present the derivations in sections 10.8–10.10.

10.3.1 Moments

The use of moments is a technique to facilitate the analysis of the effect of forces on the rotations of rigid bodies and other systems of point masses. Moments are defined as follows (see Fig. 10.2).

Definition. *Moment*
The moment vector $\vec{\boldsymbol{M}}_{B/P}$ *of a force* $\vec{\boldsymbol{F}}_B$ *(with point of action* $\vec{\boldsymbol{r}}_B$*) with respect to a reference point* $\vec{\boldsymbol{r}}_P$ *is defined as the cross product* $\vec{\boldsymbol{r}}_{B/P} \times \vec{\boldsymbol{F}}_B$*.*

$$\vec{\boldsymbol{M}}_{B/P} \equiv \vec{\boldsymbol{r}}_{B/P} \times \vec{\boldsymbol{F}}_B \quad (10.15)$$

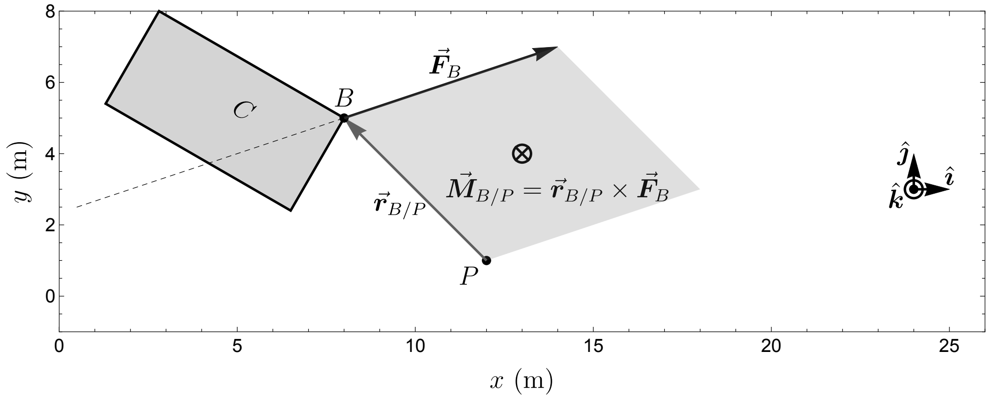


Figure 10.2: A force $\vec{F}_B$ acts on point B of rigid body C. A reference point P is chosen. The moment of force B on C with respect to P is: $\vec{M}_{B/P} = \vec{r}_{B/P} \times \vec{F}_B$. The magnitude of the moment is indicated by the area of the red parallelogram. If P lies on the line of action of $\vec{F}_B$ the moment is zero.

When using moments to analyse rigid body dynamics, for example with an FBD *it is essential that all moments are determined with respect to the same reference point P.* Furthermore, it can be shown that the sum of all internal moments on the rigid body is always zero (see Fig. 10.3), such that the resultant moment on the system is equal to the sum of all *external* moments, i.e. moments generated by external forces, as will be derived in Eq. (10.63).

Concept. *Moment on a system or rigid body*
The resultant moment vector $\vec{M}_{C/P,\text{ext}}$ acting on a system of point masses C with respect to a reference point P is the sum over the moments of the external forces with respect to P.

$$\vec{M}_{C/P,\text{ext}} = \sum_i \vec{M}_{i/P,\text{ext}} = \sum_i \vec{r}_{i/P} \times \vec{F}_{i,\text{ext}} \tag{10.16}$$

Because the sums of internal moments and forces on a system or rigid body are zero, they do not affect the kinetics. For that reason only external moments and external forces $\vec{F}_{\text{ext}}$ should be drawn in an FBD. In fact, a key purpose of an FBD is to define the difference between internal and external forces, since all objects drawn in the FBD can be considered to be *internal* and their forces should not be drawn, whereas all drawn forces are generated by external objects that are not drawn.

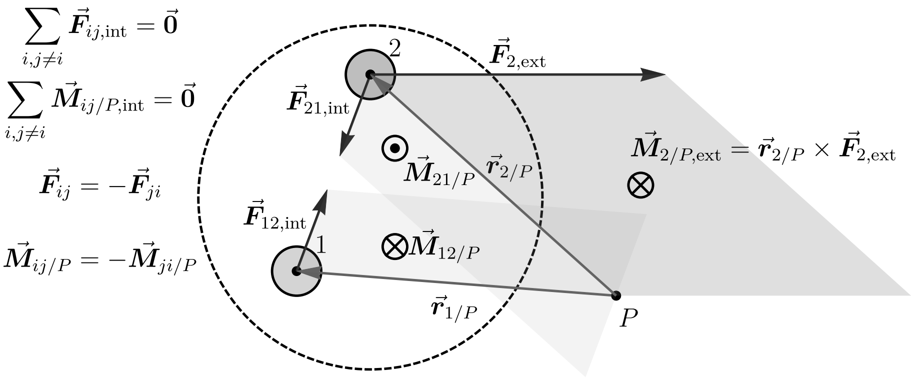


Figure 10.3: One external force $\vec{\boldsymbol{F}}_{2,\mathrm{ext}}$ acts on a system of 2 point masses that is bounded by the dashed circle. The sum of the internal forces and internal moments is always zero, independent of the choice of reference point P. So, the resultant moment on the system is equal to the sum of the external moments.

10.3.2 Angular momentum

We now define the *angular momentum* vector $\vec{\boldsymbol{L}}$ of a point mass (see Fig. 10.17), which is used to analyse rotating motion, similar to the way momentum $\vec{\boldsymbol{p}}$ is used for translational motion as described in Ch. 8.

Definition. *Angular momentum of a point mass*
The angular momentum vector $\vec{\boldsymbol{L}}_{i/P}$ of a point mass at position $\vec{\boldsymbol{r}}_i$ with momentum vector $\vec{\boldsymbol{p}}_i$, is defined as the cross product $\vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{p}}_i$.

$$\vec{\boldsymbol{L}}_{i/P} \equiv \vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{p}}_i = \vec{\boldsymbol{r}}_{i/P} \times (m_i \vec{\boldsymbol{v}}_i) \tag{10.17}$$

Concept. *Angular momentum of a system or rigid body*
The total angular momentum $\vec{\boldsymbol{L}}_{C/P}$ of a system of point masses C, with respect to reference point P is equal to the sum of the angular momentum vectors of its individual point masses.

$$\vec{\boldsymbol{L}}_{C/P} = \sum_i \vec{\boldsymbol{L}}_{i/P} = \sum_i \vec{\boldsymbol{r}}_{i/P} \times (m_i \vec{\boldsymbol{v}}_i) \tag{10.18}$$

To determine the angular momentum of a rigid body one can use this equation to sum or integrate over all point masses i in the rigid body, but a

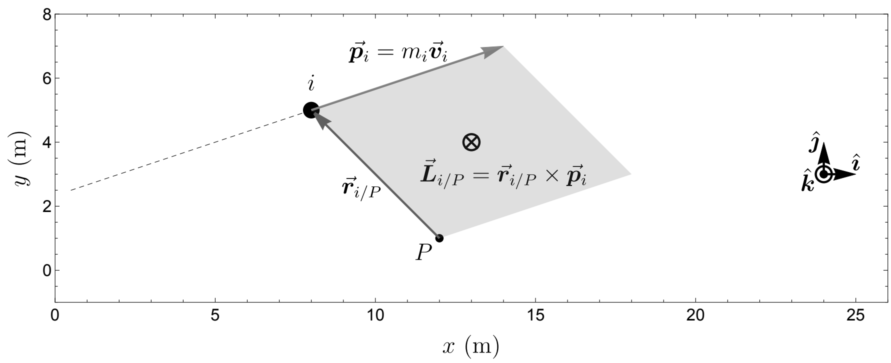


Figure 10.4: A point mass i with momentum $\vec{p}_i$ has an angular momentum $\vec{L}_{i/P} = \vec{r}_{i/P} \times \vec{p}_B$ with respect to reference point P.

simpler way is to use the moment of inertia tensor $\mathbf{I}_{C/P}$ of the rigid body or its zz component $I_{C/G,zz}$ which will be discussed in Sec. 10.5. This results in:

$$\vec{L}_{C/P} = \mathbf{I}_{C/P}\vec{\omega}_C \tag{10.19}$$

$$\vec{L}_{C/P,2D} = \left(m_{\text{tot}}\rho^2_{G/P} + I_{C/G,zz}\right)\omega_C\hat{k} \tag{10.20}$$

10.3.3 Euler's second law

We now introduce Euler's second law that will be derived in Sec. 10.10.

Concept. *Euler's second law*
Euler's second law for a system or rigid body C states that the time derivative of the total angular momentum of C is equal to the total moment from external forces acting on C. The total angular momentum should be determined with respect to the same reference point P that is fixed in an IRF or is the CoM of C.

$$\vec{M}_{C/P,\text{ext}} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{L}_{C/P} \tag{10.21}$$

Euler's first law, and Euler's second law together provide the full equations of motion for a rigid body and are therefore very useful and important for analysing their dynamics. For planar kinetics of rigid bodies Euler's second law can be simplified:

Concept. *Euler's second law for planar 2D kinetics of rigid body C*

$$M_{C/P,\text{ext},2D}\hat{k} = \vec{r}_{G/P} \times (m_{\text{tot}}\vec{a}_G) + I_{C/G,zz}\alpha_C\hat{k} \tag{10.22}$$

10.4 Solving the planar EoM for rigid bodies

After having discussed Euler's second law, we will now show how it is used to predict rotational motion of rigid bodies. Together with Euler's first law ($\sum_i \vec{\boldsymbol{F}}_{i,\text{ext}} = m_{\text{tot}}\vec{\boldsymbol{a}}_G$, Eq. (8.11)), it provides three scalar equations of motion (EoM) to analyse the planar dynamics of rigid bodies.

With Euler's laws, the translation and rotation of rigid bodies can be solved for, using very similar methods as were outlined in Sec. 6.4 for the kinetic analysis of point masses. The main novel aspect is the determination and solution of the rotational EoM that follows from Euler's second law, including the moments and angular accelerations. Let us summarise the procedure for solving the equations of motion for the planar kinetics of a rigid body, step-by-step:

1. Sketch the rigid bodies, massless mechanisms, force vectors, moments, distances, dimensions and constraints.
2. Choose and draw a suitable coordinate system (CS) and reference point P for moments.
3. Determine the constraint equations.
4. Determine the CoM $\vec{\boldsymbol{r}}_G$ and moment of inertia I_G for each rigid body.
5. Draw the free-body diagram (FBD) for each rigid body. Add both forces and moments[1].
6. Project force, moment and kinematic vectors onto the CS to obtain scalar components (like F_x, $a_{G,x}$, M_z and α_z).
7. Determine the 3 equations of motion per rigid body. Two equations, for $a_{G,x}$ and $a_{G,y}$ from Euler's first law, and one from Euler's second law to determine α_z.
8. Simplify the EoM by combining them and using constraint equations.
9. Solve the simplified (differential) equations of motion (EoM), determining velocity and motion using integration or other methods.

This procedure follows to a large extent the methodologies for point masses that were discussed in Ch. 6. Nevertheless, there are some new aspects that need to be considered for rotating rigid bodies like:

- Selecting the easiest point of reference P for analysing the problem. In principle every point can be chosen, but the analysis can be facilitated if it allows simplifying moment or kinematic expressions (Sec. 9.5.3). The

[1] Don't draw items double, only draw moments explicitly if they are not represented as a force.

choice of the point P will not affect the dynamics, a smart choice can however simplify the calculation.

- For Euler's second law to hold, the reference point P should be a fixed point in an IRF, any fixed point P in an IRF can be chosen. Alternatively the point G that moves along with the CoM of the rigid body, or system of point masses, can be chosen.
- It is important to use the same reference point P both for the moment and the angular momentum.
- Often, one first needs to use Euler's first law and solve the corresponding EoMs to determine $\vec{\boldsymbol{a}}_G$, before the term $\vec{\boldsymbol{r}}_{G/P} \times (m\vec{\boldsymbol{a}}_G)$ in Euler's second law Eq. (10.22) can be evaluated to solve for the angular acceleration.

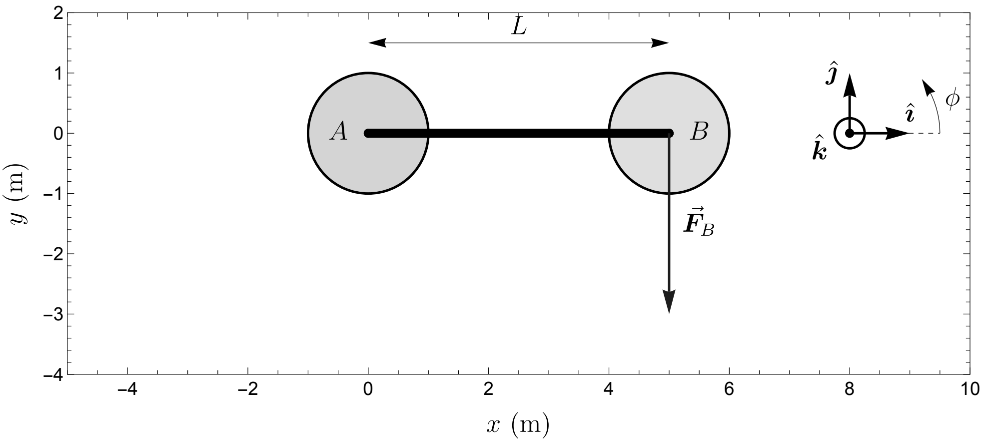


Figure 10.5: Sketch of a rigid body C on which a force $\vec{\boldsymbol{F}}_B$ acts.

S **Example 10.2** We are now going to solve the problem from Example 10.1 using Euler's laws.

- *Determine the angular acceleration of the dumbbell-shaped rigid body C in Fig. 10.5.*

We assume that the two masses are identical $m_A = m_B = m$ and follow the steps introduced in this chapter.

- Step 1 and 2: First we use the sketch in Fig. 10.5 and also indicate the Cartesian coordinate system and corresponding unit vectors in it.
- Step 3 and 4: The motion is unconstrained and the equation for the CoM is:

$$\vec{\boldsymbol{r}}_G = \frac{m_A\vec{\boldsymbol{r}}_A + m_B\vec{\boldsymbol{r}}_B}{m_A + m_B} = \frac{m \cdot 0 + mL\hat{\boldsymbol{\imath}}}{2m} = \frac{1}{2}L\hat{\boldsymbol{\imath}} \tag{10.23}$$

The moment of inertia with respect to the CoM can be shown to be $I_G = \frac{1}{2}mL^2$. We choose the CoM as the reference point for rotations, so $P = G$.

- Step 5: Now we draw the FBD in Fig. 10.6 according to the guidelines given in Sec. 6.8. It is important not to draw it in the 'trivial' horizontal position, but at an angle ϕ.
- Step 6: We project the forces, moments and kinematic vectors: $\vec{\boldsymbol{F}}_B = -F_B\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{M}}_{B/G} = \vec{\boldsymbol{r}}_{B/G} \times \vec{\boldsymbol{F}}_B = -r_{B/G,x}F_B\hat{\boldsymbol{k}} = -\frac{L}{2}\cos(\phi)F_B\hat{\boldsymbol{k}}$. Furthermore we have $\vec{\boldsymbol{a}}_G = a_{G,x}\hat{\boldsymbol{\imath}} + a_{G,y}\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{\alpha}}_C = \alpha_C\hat{\boldsymbol{k}}$.
- Step 7: We first apply Euler's first law $\sum \vec{\boldsymbol{F}} = m_{\text{tot}}\vec{\boldsymbol{a}}_G$.

$$\sum F_x = 0 \quad = \quad 2ma_{G,x} \tag{10.24}$$

$$\sum F_y = -F_B \quad = \quad 2ma_{G,y} \tag{10.25}$$

Now we apply Euler's second law $M_G\hat{\boldsymbol{k}} = \vec{\boldsymbol{r}}_{G/P} \times (m_{\text{tot}}\vec{\boldsymbol{a}}_G) + I_G\alpha\hat{\boldsymbol{k}}$. Since $P = G$ the vector $\vec{\boldsymbol{r}}_{G/G} = \vec{\boldsymbol{0}}$. Thus we have:

$$-\frac{L}{2}\cos(\phi)F_B\hat{\boldsymbol{k}} = I_G\alpha_C\hat{\boldsymbol{k}} \tag{10.26}$$

- Step 8: By rewriting the last 3 equations, the simplified EoMs become:

$$a_{G,x} = \ddot{x}_G \quad = \quad 0 \tag{10.27}$$

$$a_{G,y} = \ddot{y}_G \quad = \quad -F_B/(2m) \tag{10.28}$$

$$\alpha_C = \ddot{\phi} \quad = \quad \frac{-\frac{L}{2}\cos(\phi)F_B}{\frac{1}{2}mL^2} = \frac{-F_B}{mL}, \tag{10.29}$$

where we used in the last step that $I_G = \frac{1}{2}mL^2$ and $\phi = 0$ in our case. Note that the result for α_C is identical to that in Eq. (10.11), as expected.

- Step 9: In step 8 we have determined α_C as requested. To obtain the full time dependence of $x_G(t)$, $y_G(t)$ and $\phi_C(t)$ we would have to solve the EoMs differential equations.

10.5 Moment of inertia

10.5.1 Inertia tensor

In Sec. 10.3.2 we have introduced the moment of inertia to simplify the description of the angular momentum of rigid bodies. Let's discuss it in more detail now.

Definition. *Inertia tensor*
The inertia tensor $\mathbf{I}_G$ *of a rigid body with respect to its CoM is defined by the*

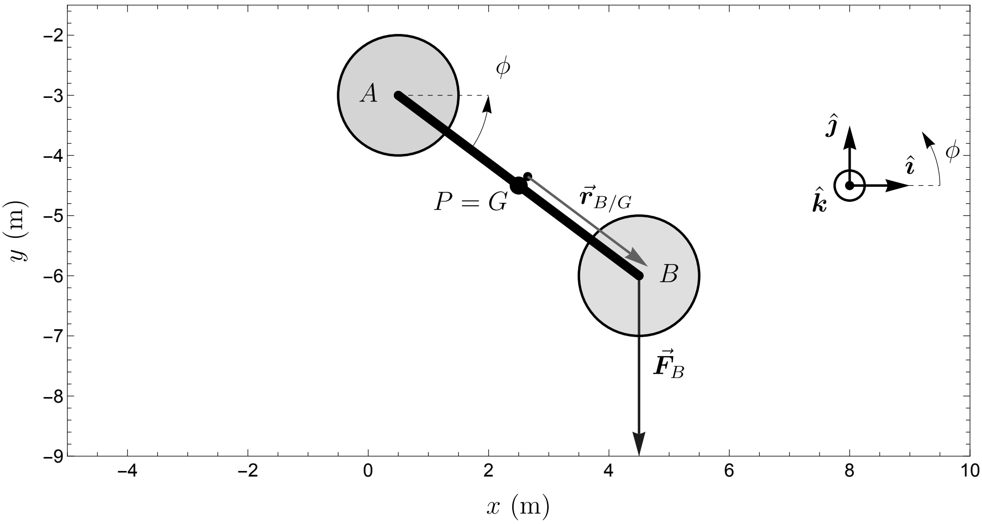


Figure 10.6: FBD of a rigid body C on which a force $\vec{\boldsymbol{F}}_B$ acts.

equation:

$$\vec{\boldsymbol{L}}_G = \mathbf{I}_G\vec{\boldsymbol{\omega}} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \vec{\boldsymbol{\omega}} \tag{10.30}$$

The diagonal components of the inertia tensor are called the moments of inertia, and as will be derived later in Eq. (10.79) they can be calculated as follows:

$$I_{G,zz} = \sum_i m_i \rho_{i/G}^2 \tag{10.31}$$

The off-diagonal terms of the inertia tensor are called products of inertia. As an example $I_{G,zz}$ can be written as:

$$I_{G,xz} = -\sum_i m_i x_{i/G} z_{i/G} \tag{10.32}$$

It can be seen that if the rigid body has mirror symmetry in the $z_{i/G} = 0$ plane, or if it is axisymmetric around the z-axis, the products of inertia $I_{G,zx}$ and $I_{G,zy}$ are zero. E.g. because the contributions for $z_{i/G}$ and $-z_{i/G}$ cancel each other. In this textbook we focus only on rigid bodies with either of those properties, such that we only need to calculate the moment of inertia I_{zz}.

10.5.2 Calculation of the moment of inertia

The moment inertia of a rigid body can be calculated by summing the contribution $m_i\rho^2_{i/G}$ of every point mass i, however often we are dealing with continuous materials with a constant mass density ρ_m in kg/m^3. In those cases the body can be subdivided into infinitesimal volumes dV with mass d$m = \rho_m$dV and contribution to the moment of inertia $\rho^2_{i/G}$dm. The moment of inertia can then be calculated by integration over the total volume V of the rigid body.

$$I_{G,zz} = \sum_i m_i\rho^2_{i/G} = \int_V \rho_m\rho^2_{i/G}\mathrm{d}V \tag{10.33}$$

Note that $\rho_{i/G}$ is the distance of a point i from the axis parallel to the z-axis that goes through G. Let us give a few examples of how this integral can be performed, where the CoM is positioned in the origin of the coordinate system.

S **Example 10.3** *Moment of inertia of a block*

For a block with dimensions L_x, L_y, L_z along the three axes, we use $\rho^2 = x^2 + y^2$ to determine its moment of inertia by integrating over its volume along each of the three axes:

$$I_{G,zz} = \rho_m \int_{-L_y/2}^{L_y/2} \mathrm{d}y \int_{-L_x/2}^{L_x/2} (x^2+y^2)\mathrm{d}x \int_{-L_z/2}^{L_z/2} \mathrm{d}z \tag{10.34}$$

$$= \rho_m L_z \int_{-L_y/2}^{L_y/2} \left[\frac{1}{3}x^3 + y^2x\right]_{-L_x/2}^{L_x/2} \mathrm{d}y \tag{10.35}$$

$$= \rho_m L_z \left[\frac{1}{12}L_x^3 y + \frac{1}{3}y^3 L_x\right]_{-L_y/2}^{L_y/2} \tag{10.36}$$

$$= \frac{1}{12}\rho_m(L_zL_xL_y)\left[L_x^2 + L_y^2\right] = \frac{1}{12}m\left[L_x^2 + L_y^2\right] \tag{10.37}$$

Note that the triple integral is solved in 3 steps, starting with the innermost integral going from right to left, first integrate over z, then over x and finally over y. Multiplicative terms over which we don't integrate can be moved to the left, out of the integral. In the last step we used that mass is density times volume $m = \rho_m L_z L_x L_y$.

Moment of inertia of axisymmetric bodies

When an object is axisymmetric, it is easiest to determine its moment of inertia by using cylindrical coordinates. An infinitesimal volume element in these coordinates has a volume d$V = \rho$dϕdρdz, where the size in the $\hat{\boldsymbol{\phi}}$ direction is given by the infinitesimal circle segment ρdϕ. The outer surface of an axisymmetric object can be described by a function $\rho_d(z)$ or alternatively by $z_h(\rho)$. We will show two examples to illustrate integration of $\rho_m \rho_{i/G}$ over either volume to obtain the moment of inertia of a cone with radius R and total height h.

S **Example 10.4** *Moment of inertia of a cone method 1*
The height of the cone at each ρ can be described by $z_h(\rho) = h(1 - \frac{\rho}{R})$, such that I_{zz} becomes:

$$\begin{aligned} I_{G,zz} &= \rho_m \int_0^R \int_0^{z_h(\rho)} \rho^2 \int_0^{2\pi} \rho \mathrm{d}\phi \mathrm{d}\rho \mathrm{d}z = 2\pi\rho_m \int_0^R \rho^3 [z]_0^{h\left(1-\frac{\rho}{R}\right)} \mathrm{d}\rho \\ &= 2\pi\rho_m \int_0^R \rho^3 h \left(1 - \frac{\rho}{R}\right) \mathrm{d}\rho = 2\pi\rho_m h \left[\frac{1}{4}\rho^4 - \frac{1}{5R}\rho^5\right]_0^R \\ &= \frac{2}{20}\pi\rho_m h R^4 = \frac{3}{10} m R^2, \end{aligned} \tag{10.38}$$

where we used in the last step that the volume of a cone is $V = \frac{h}{3}\pi R^2$.

S **Example 10.5** *Moment of inertia of a cone method 2* Alternatively, the radius of the disks of the cone depends on height as $\rho_d(z) = R(1 - \frac{z}{h})$. We now again perform the integral:

$$\begin{aligned} I_{G,zz} &= \rho_m \int_0^h \int_0^{\rho_d(z)} \rho^2 \int_0^{2\pi} \rho \mathrm{d}\phi \mathrm{d}\rho \mathrm{d}z = 2\pi\rho_m \int_0^h \int_0^{R(1-\frac{z}{h})} \rho^3 \mathrm{d}\rho \mathrm{d}z \\ &= 2\pi\rho_m \int_0^h \left[\frac{1}{4}\rho^4\right]_0^{R(1-\frac{z}{h})} \mathrm{d}z = \frac{1}{2}\pi\rho_m \frac{R^4}{h^4} \int_0^h (h-z)^4 \mathrm{d}z \\ &= \frac{1}{2}\pi\rho_m \frac{R^4}{h^4} \left[-\frac{1}{5}(h-z)^5\right]_0^h \mathrm{d}z \qquad (10.39) \\ &= \frac{1}{10}\pi\rho_m h R^4 = \frac{3}{10} m R^2 \qquad (10.40) \end{aligned}$$

We obtain the same result, as expected. Determining the moment of inertia of rigid bodies is a good practice in volume integration skills.

10.5.3 Parallel axis theorem

We have now a method to calculate the moment of inertia of a rigid body with respect to its CoM. But often a reference point P is chosen that is different from G. In that case one can redo the integrals again and obtain $I_{P,zz}$, but there is an easier way to determine $I_{P,zz}$ from $I_{G,zz}$, because as will be shown in Eq. (10.72), the angular momentum with respect to a point P can be written as the sum of two parts, namely a component representing the angular momentum a point mass m_{tot} at position G and a component representing the angular momentum with respect to G:

$$\vec{\boldsymbol{L}}_{C/P} = \vec{\boldsymbol{L}}_{G/P} + \sum_i \vec{\boldsymbol{L}}_{i/G} \tag{10.41}$$

$$\vec{\boldsymbol{L}}_{C/P,2D} = \left(m_{\text{tot}}\rho^2_{G/P} + I_{G,zz}\right)\vec{\boldsymbol{\omega}}_C \tag{10.42}$$

$$\vec{\boldsymbol{L}}_{C/P,2D} = I_{P,zz}\vec{\boldsymbol{\omega}}_C \tag{10.43}$$

Now from the last two equations we can determine $I_{P,zz}$, this method is called the parallel axis theorem or Steiner's theorem.

Concept. *Parallel axis theorem*
The moment of inertia of a rigid body around an axis through a point P, at a distance $\rho_{G/P}$ from its CoM, equals the moment of inertia $m_{\text{tot}}\rho^2_{G/P}$ of its CoM, plus the moment of inertia of the rigid body around a parallel axis through its CoM.

$$I_{zz,P} = m_{\text{tot}}\rho^2_{G/P} + I_{G,zz} \tag{10.44}$$

S **Example 10.6** *Determine the moment inertia $I_{zz,P}$ of the rectangular rigid body with mass m with respect to reference point P at the origin.*
Since the rigid body lies in the xy-plane we have $\rho_{G/P} = |\vec{\boldsymbol{r}}_{G/P}|$. The moment of inertia $I_{G,zz}$ of a rectangle can be determined using Eq. (10.37). Then with the parallel axis theorem we find:

$$I_{P,zz} = m\rho^2_{G/P} + \frac{1}{12}m\left(L_x^2 + L_y^2\right) \tag{10.45}$$

10.5.4 Adding and subtracting moments of inertia

Since the moment of inertia is a sum or integral over individual contributions for every point mass, the moment of inertia of a rigid body C that is made up out of two (or more) other rigid bodies A and B can be calculated by summing the moments of inertia around the same axis P:

$$I_{C,P,zz} = I_{A,P,zz} + I_{B,P,zz} \tag{10.46}$$

Make sure you use the same reference point P when making such additions. Similarly, it is also possible to subtract moments of inertia from one another, for instance to determine the moment of inertia of objects that have a hole.

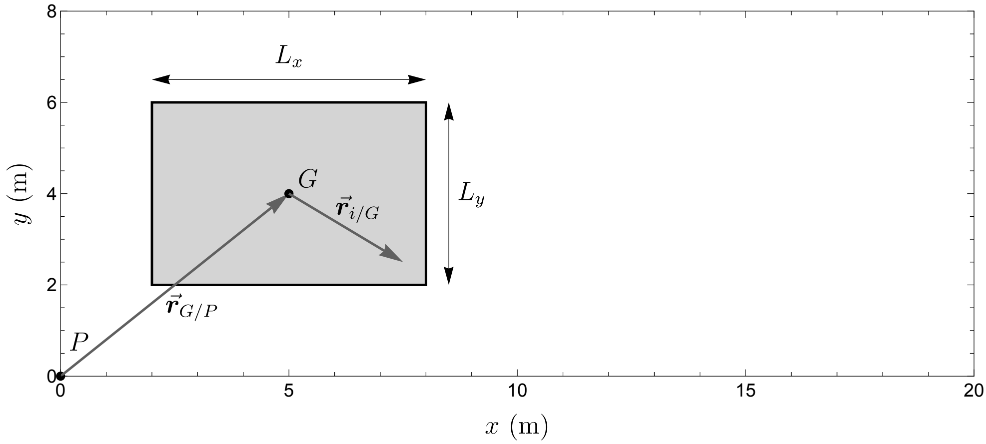


Figure 10.7: With the parallel axis theorem the moment of inertia with respect to an arbitrary axis P that is not the CoM of the rigid body can be determined.

10.6 Euler's second law for special cases

We now discuss a few special types of motion that can be analysed with Euler's second law.

10.6.1 Pure translation

In the case of pure translation, we have no angular acceleration $\vec{\boldsymbol{\alpha}} = \vec{\mathbf{0}}$, which is the case if the moment of all external forces with respect to the CoM is zero ($\vec{\boldsymbol{M}}_{C,G} = \vec{\mathbf{0}}$). This situation is for instance valid for the moments generated in a constant gravitational field, or when constraint equations prevent rotations. Then Eq. (10.22) becomes:

$$\begin{aligned} \text{If}: \sum_i \vec{\boldsymbol{M}}_{C,i/G} &= I_{G,zz}\alpha_C \hat{\boldsymbol{k}} = \vec{\mathbf{0}} \qquad (10.47) \\ \sum_i M_{C,i/P,2D}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{r}}_{G/P} \times (m_{\text{tot}}\vec{\boldsymbol{a}}_G) = \vec{\boldsymbol{r}}_{G/P} \times \left(\sum_i \vec{\boldsymbol{F}}_{i,\text{ext}}\right) \end{aligned}$$

It can be seen that once it is known that $\vec{\boldsymbol{\alpha}}_C = \vec{\mathbf{0}}$, Euler's second law on the moments does not provide additional useful information beyond Euler's first law that determines the acceleration of the CoM of the rigid body via $\sum_i \vec{\boldsymbol{F}}_{i,\text{ext}} = m_{\text{tot}}\vec{\boldsymbol{a}}_G$.

10.6.2 Pure rotation around the centre of mass

It is interesting to consider the situation where the term $m_{\text{tot}}\vec{\boldsymbol{a}}_G = \vec{\mathbf{0}}$ is zero because the sum of forces on the rigid body is zero, since it significantly simplifies Eq. (10.22), with the disappearance of the term $\vec{\boldsymbol{r}}_{G/P} \times (m_{\text{tot}}\vec{\boldsymbol{a}}_G)$.

$$\text{If}: \sum_i \vec{\boldsymbol{F}}_{\text{i,ext}} = m_{\text{tot}}\vec{\boldsymbol{a}}_G = \vec{\mathbf{0}} \tag{10.48}$$

$$\text{Then}: \sum_i M_{C,i/P,2D}\hat{\boldsymbol{k}} = I_{G,zz}\alpha_C\hat{\boldsymbol{k}} \tag{10.49}$$

We thus find in Eq. (10.49) that the CoM of the rigid body does not accelerate and the acting moments purely cause an angular acceleration α_C of the rigid body.

10.6.3 Couple

We saw in the previous subsection that a special situation arises if the sum of the forces acting on a system is zero, since then $\vec{\boldsymbol{a}}_G = \vec{\mathbf{0}}$. Such a combination of forces is called a *couple*, and if there are only two forces it is called a *simple couple*.

Concept. *Simple couple*
A simple couple consists of two forces $\vec{\boldsymbol{F}}_A = -\vec{\boldsymbol{F}}_B$ that are equal in magnitude, and opposite in direction such that $\vec{\boldsymbol{F}}_A + \vec{\boldsymbol{F}}_B = \vec{\mathbf{0}}$, while the forces have different points of action, such that their resultant moment is not zero.

Besides simple couples, that consist of two forces, there is also a more general definition of a couple, that can consist of more than two forces.

Concept. *Couple*
A couple consists of a number of N forces for which the sum of forces is zero, that generate a resultant moment on a system. So $\sum_{i=1}^{N} \vec{\boldsymbol{F}}_{i,\text{ext}} = \vec{\mathbf{0}}$.

According to Euler's second law, a couple only influences the angular acceleration α of a rigid body, it does not contribute to the acceleration $\vec{\boldsymbol{a}}_G$ of the CoM because the sum of the forces is zero. The effect of a couple is therefore also called a *pure moment* or *couple moment*.

Another useful property of a couple is that its resultant pure moment is independent of the choice of the location of the reference point P, as we will derive now.

Derivation. *Moment of a couple*
Consider a simple couple generated by two force vectors with $\vec{\boldsymbol{F}}_A = -\vec{\boldsymbol{F}}_B$ *acting on points* A *and* B*. The moment of this couple with respect to a reference point* P *is given by:*

$$
\begin{aligned}
\vec{\boldsymbol{M}}_{\text{couple}} &= \vec{\boldsymbol{r}}_{A/P} \times \vec{\boldsymbol{F}}_A + \vec{\boldsymbol{r}}_{B/P} \times \vec{\boldsymbol{F}}_B && (10.50)\\
&= \vec{\boldsymbol{r}}_{A/P} \times \vec{\boldsymbol{F}}_A - \vec{\boldsymbol{r}}_{B/P} \times \vec{\boldsymbol{F}}_A && (10.51)\\
&= ((\vec{\boldsymbol{r}}_A - \vec{\boldsymbol{r}}_P) - (\vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_P)) \times \vec{\boldsymbol{F}}_A && (10.52)\\
&= \vec{\boldsymbol{r}}_{A/B} \times \vec{\boldsymbol{F}}_A && (10.53)
\end{aligned}
$$

From this equation we see that the couple vector $\vec{\boldsymbol{M}}_{\text{couple}}$ is independent of the choice of the reference point P and only depends on the force vectors of the couple and their points of action $\vec{\boldsymbol{r}}_A$ and $\vec{\boldsymbol{r}}_B$.

10.6.4 Pure rotation around a fixed axis

Let us now consider pure rotation of a rigid body C about a fixed axis P parallel to the z-axis, which does not translate ($\vec{\boldsymbol{a}}_P = \vec{\boldsymbol{v}}_P = \vec{\mathbf{0}}$). We choose point P as reference point. There is no translation of this point, only the angular acceleration of the rigid body α. As a consequence we have from Eq. (9.29) that $\vec{\boldsymbol{a}}_G = \vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{G/P} - \omega^2 \vec{\boldsymbol{r}}_{G/P}$. Substituting this into Eq. (10.22) we obtain:

$$
\begin{aligned}
& \text{If: } \vec{\boldsymbol{a}}_P = \vec{\boldsymbol{v}}_P = \vec{\mathbf{0}} && (10.54)\\
\sum_i M_{C,i/P,2D} \hat{\boldsymbol{k}} &= m_{\text{tot}} \vec{\boldsymbol{r}}_{G/P} \times (\vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{r}}_{G/P} - \omega^2 \vec{\boldsymbol{r}}_{G/P}) + I_{G,zz} \alpha_C \hat{\boldsymbol{k}} \\
&= m_{\text{tot}} \rho_{G/P} \hat{\boldsymbol{\rho}} \times (\alpha \hat{\boldsymbol{k}} \times \rho_{G/P} \hat{\boldsymbol{\rho}}) + I_{G,zz} \alpha_C \hat{\boldsymbol{k}} \\
&= (m_{\text{tot}} \rho_{G/P}^2 + I_{G,zz}) \alpha \hat{\boldsymbol{k}} && (10.55)\\
&= I_{P,zz} \alpha \hat{\boldsymbol{k}} && (10.56)
\end{aligned}
$$

In the last step we used the parallel axes theorem $I_{zz,P} = m_{\text{tot}} \rho_{G/P}^2 + I_{G,zz}$, where $\rho_{G/P}$ is the smallest distance between the parallel axes through G and P. The conclusion from Eq. (10.56) is that for a rigid body C that rotates in-plane around a fixed axis P: the sum of the moments with respect to the fixed axis P equals the product of the moment of inertia $I_{zz,P}$ and the angular acceleration α_C of the rigid body C.

10.7 Vectors in dynamics

In this textbook many different kinetic and kinematic vectors are introduced. We summarise these vectors, their labelling and their dependence on reference points.

10.7.1 Dependencies of vectors on reference points

When a single force $\vec{\boldsymbol{F}}_A$ acts on a rigid body C with point of action $\vec{\boldsymbol{r}}_A$ and reference point $\vec{\boldsymbol{r}}_P$ for rotations, a variety of vectors are used to analyse the dynamics. These vectors and their dependence on position vectors $\vec{\boldsymbol{r}}_A$ and $\vec{\boldsymbol{r}}_P$ are summarised in Table 10.1.

Table 10.1: Dependencies of the vectors used to analyse the dynamics of a rigid body C with CoM G, on the point of action A of force $\vec{\boldsymbol{F}}_A$ and the choice of reference point P.

Vector	Depends on:	
	$\vec{\boldsymbol{r}}_A$	$\vec{\boldsymbol{r}}_P$
$\vec{\boldsymbol{F}}_A$	✓	✗
$\vec{\boldsymbol{a}}_G$	✗	✗
$\vec{\boldsymbol{M}}_{A/P}$	✓	✓
$\vec{\boldsymbol{M}}_{\text{couple}}$	$\vec{\boldsymbol{r}}_{A/B}$	✗
$\vec{\boldsymbol{\alpha}}_C$	✓	✗
$\vec{\boldsymbol{L}}_{C/P}$	✗	✓
$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{L}}_{C/P}$	✓	✓

Note that the subscript P is added to the vectors $\vec{\boldsymbol{M}}_{A/P}$ and $\vec{\boldsymbol{L}}_{C/P}$ because they depend on $\vec{\boldsymbol{r}}_P$. It might seem strange that a moment vector can depend on the choice of the reference point P, while the dynamics of the system (i.e. the acceleration $\vec{\boldsymbol{a}}_G$ and angular acceleration $\vec{\boldsymbol{\alpha}}_C$) do not depend on it. To understand this, we repeat Euler's second law for planar motion Eq. (10.22):

$$M_{C/P,2D}\hat{\boldsymbol{k}} = \vec{\boldsymbol{r}}_{G/P} \times (m_{\text{tot}}\vec{\boldsymbol{a}}_G) + I_{C/G,zz}\alpha_C\hat{\boldsymbol{k}} \tag{10.57}$$

It can be seen that changing P only results in a change in the relative position $\vec{\boldsymbol{r}}_{G/P}$, such that even for identical dynamics ($\vec{\boldsymbol{a}}_G$ and α_C both constant), the moment vector $\vec{\boldsymbol{M}}_{C/P,2D}$ acting on the system C can change. Another way to look at is to realise that Eq. (10.22) shows that a moment vector $\vec{\boldsymbol{M}}_{C/P}$, that originates from a force $\vec{\boldsymbol{F}}_A = m_{\text{tot}}\vec{\boldsymbol{a}}_G$, has two effects: 1. Linear acceleration of the CoM of the system, such that the angular momentum $\vec{\boldsymbol{L}}_{C/P}$ of G with respect to P changes. 2. Generate an angular acceleration $\vec{\boldsymbol{\alpha}}_C$. Only the first effect depends on the choice of P. Remember that the reference point P should either be chosen as a fixed point in an IRF, or as the (accelerating) point $P = G$ that moves along with the CoM of the system C. It is *not* allowed to choose another accelerating or rotating point as reference point P, since then Euler's second law does not hold anymore. Finally note that the dependence of $\vec{\boldsymbol{\alpha}}_C$ and $\frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{L}}_{C/P}$ on the point of action A of the force in Table 10.1 is not a direct dependence, since you can calculate these vectors from kinematics without knowing what A is (there is only a dependence on the point of action via Euler's second law), therefore A is not needed as subscript in these cases.

10.7.2 Labelling scalars and vectors in dynamics

As discussed in the previous subsection, the magnitude and direction of vectors can depend on the point of reference and point of action, and therefore these points have to be clearly identified to prevent making mistakes. Here we will describe how to provide this kind of important information via labels (see also Sec. 2.1.2). Labelling can save time, since it avoids having to writing for each symbol a sentence like: "$\vec{\boldsymbol{M}}_{A/P}$ is the moment vector generated by force $\vec{\boldsymbol{F}}_A$ (with point of action A) with respect to reference point P". Labels reduce the risk that confusion can occur about the meaning of a symbol. Here we provide a few conventions we recommend for using labels in dynamics:

1. Every mathematical quantity (e.g. scalar, vector, tensor) should get a unique symbol to identify it, conventions are:
   - $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{a}}$ are symbols for position, velocity and acceleration vectors.
   - $\vec{\boldsymbol{\omega}}$, $\vec{\boldsymbol{\alpha}}$ are symbols for angular velocity and acceleration vectors.
   - m, I, $\vec{\boldsymbol{p}}$ and $\vec{\boldsymbol{L}}$ are mass, moment of inertia, momentum and angular momentum vectors.
   - $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{M}}$ are force and moment vectors (so e.g. don't use $\vec{\boldsymbol{N}}$ for a normal force but instead use $\vec{\boldsymbol{F}}_{\mathrm{N}}$).
   - W, V and T are work, potential energy and kinetic energy.
   - $\vec{\boldsymbol{J}}$ and $\vec{\boldsymbol{H}}$ are impulse and angular impulse.
2. If the problem deals with multiple objects, each object should get a unique label (letter, number, word), that is indicated in the sketch.
3. Every relevant point (e.g. position, CoM, point of action, reference point) in the sketch should get a unique label.
4. Every quantity should get sufficient subscripts to define it uniquely. We will use the following conventions for these definitions:
   - $\vec{\boldsymbol{r}}_A$, $\vec{\boldsymbol{v}}_A$, $\vec{\boldsymbol{a}}_A$ get a subscript A to indicate the point or point mass A to which they refer.

- $\vec{\boldsymbol{\omega}}_C$, $\vec{\boldsymbol{\alpha}}_C$ get subscripts C to indicate the system/rigid body C they refer to.
- Each mass m_C gets a subscript to identify the object C it refers to, while a moment of inertia $I_{C/P,zz}$ also refers to the reference point P. The component zz of the inertia tensor can be left out if it is clear that one deals with planar motion in the xy-plane.
- $\vec{\boldsymbol{p}}_C = m_C \vec{\boldsymbol{v}}_{C,G}$ is the momentum of object C with total mass m_C and $\vec{\boldsymbol{L}}_{C/P}$ is its angular momentum vector with respect to reference point P.
- $\vec{\boldsymbol{F}}_A$ is a force that acts on point of action A. A label to indicate the source of the force can be added, like $\vec{\boldsymbol{F}}_{\text{rope},A}$. If two objects i and j generate forces on each other, then $\vec{\boldsymbol{F}}_{ij}$ is the force acting on i and $\vec{\boldsymbol{F}}_{ji}$ is the force acting on j, with $\vec{\boldsymbol{F}}_{ij} = -\vec{\boldsymbol{F}}_{ji}$ according to Newton's third law.
- $\vec{\boldsymbol{M}}_{A/P}$ is the moment vector as a result of the force $\vec{\boldsymbol{F}}_A$ with respect to reference point P. Note that we use the slash / with the meaning 'with respect to' or 'relative to' similar as in relative position vectors.
- $\vec{\boldsymbol{J}}_{A,12}$ is the impulse vector of a force A on the time interval between t_1 and t_2 and $\vec{\boldsymbol{H}}_{A/P,12}$ is the angular impulse vector of a force A with respect to reference point P between t_1 and t_2.
- $W_{A,12}$ is the work done by a force A between two states (times or positions), V_A is the potential energy of a force (field) A and T_C is the kinetic energy of an object C.

5. In some cases quantities can be labelled uniquely with fewer subscripts. E.g. if there is only one object with mass in the problem, that mass can be given the label m without subscript, but it is never allowed to leave out the reference point labels P for moments $\vec{\boldsymbol{M}}_{A/P}$, moments of inertia $I_{P,zz}$, angular momentum $\vec{\boldsymbol{L}}_P$, and impulse vectors $\vec{\boldsymbol{H}}_{A/P,12}$, since the whole meaning of those quantities is tied to the choice of reference point P.
6. The use of labels according to the conventions above is a replacement for clearly defining every symbol in words or equations separately. Providing such a definition is also a valid approach.

10.8 Derivation moment on a system*

In this section we will derive and prove that the resultant moment of internal forces on a system of point masses is zero, such that we only need to add the moments of external forces to analyse the kinetics with Euler's second law.

Derivation. *Resultant moment on a system of many point masses.*

If many forces are acting on a system C consisting of many point masses, we can calculate the resultant moment $\vec{\boldsymbol{M}}_{C,P}$ on the system with respect to a single reference point P, by adding Eq. (10.15) for all point masses.

$$\vec{\boldsymbol{M}}_{C,P} = \sum_i \vec{\boldsymbol{M}}_{i/P} = \sum_i \vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_i \tag{10.58}$$

Note that we always use the same reference point P for every moment in the sum. The forces acting can be separated into external and internal forces, where each point mass i experiences a moment due to the sum of external forces $\vec{\boldsymbol{F}}_{i,\text{ext}}$ and due to the internal force vectors $\vec{\boldsymbol{F}}_{ij}$ acting on it by the other point masses for which index $j \neq i$:

$$\vec{\boldsymbol{M}}_{C,P} = \sum_i \vec{\boldsymbol{M}}_{i/P} = \sum_i \vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_{i,\text{ext}} + \sum_i \sum_{j \neq i} \vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_{ij} \tag{10.59}$$

We know from Newton's third law that the internal forces always come in pairs, such that we have both moments $\vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_{ij}$ and $\vec{\boldsymbol{r}}_{j/P} \times \vec{\boldsymbol{F}}_{ji}$ contributing to $\vec{\boldsymbol{M}}_{C,P}$. When adding these, as illustrated for $\vec{\boldsymbol{F}}_{12,\text{int}}$ and $\vec{\boldsymbol{F}}_{21,\text{int}}$ in Fig. 10.3, we get:

$$\vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_{ij} + \vec{\boldsymbol{r}}_{j/P} \times \vec{\boldsymbol{F}}_{ji} = \tag{10.60}$$

$$\vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_{ij} + \vec{\boldsymbol{r}}_{j/P} \times (-\vec{\boldsymbol{F}}_{ij}) = \tag{10.61}$$

$$(\vec{\boldsymbol{r}}_{i/P} - \vec{\boldsymbol{r}}_{j/P}) \times \vec{\boldsymbol{F}}_{ij} = \tag{10.62}$$

$$\vec{\boldsymbol{r}}_{i/j} \times \vec{\boldsymbol{F}}_{ij} = \vec{\boldsymbol{0}} \tag{10.63}$$

Where we used that $\vec{\boldsymbol{F}}_{ji} = -\vec{\boldsymbol{F}}_{ji}$ from Newton's third law and use that $\vec{\boldsymbol{r}}_{i/j} \times \vec{\boldsymbol{F}}_{ij} = \vec{\boldsymbol{0}}$ because of the collinearity of the forces along the line connecting the point masses, that is parallel to $\vec{\boldsymbol{r}}_{i/j}$. From Eq. (10.63) we conclude that every contribution to the resultant moment by an internal force $\vec{\boldsymbol{F}}_{ij}$ is cancelled by another contribution by a force $\vec{\boldsymbol{F}}_{ji}$. Therefore, the internal forces do not contribute to the resultant moment acting on the system and Eq. (10.59) can be simplified by only considering external forces.

Concept. *Moment on a system*
The resultant moment vector $\sum_i \vec{\boldsymbol{M}}_{C,i/P}$ *acting on a system of point masses* C *with respect to a reference point* P *is the sum over the moments of the external forces.*

$$\sum_i \vec{\boldsymbol{M}}_{C,i/P,\mathrm{ext}} = \sum_i \vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_{i,\mathrm{ext}} \tag{10.64}$$

10.9 Angular momentum of a rigid body*

10.9.1 Angular momentum of a point mass

Similar to the momentum $\vec{\boldsymbol{p}}_i = m_i \vec{\boldsymbol{v}}_i$ (which is also called the linear momentum), we defined in Eq. (10.17) the angular momentum of a point mass as:

$$\vec{\boldsymbol{L}}_{i/P} \equiv \vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{p}}_i = \vec{\boldsymbol{r}}_{i/P} \times (m_i \vec{\boldsymbol{v}}_i) \tag{10.65}$$

10.9.2 Angular momentum of a system

Using Eq. (10.65), we can now determine the angular momentum of a system of point masses, by summing the angular momentum of all point masses, for the same reference point P:

$$\vec{\boldsymbol{L}}_{C,P} = \sum_i \vec{\boldsymbol{L}}_{i/P} = \sum_i \vec{\boldsymbol{r}}_{i/P} \times (m_i \vec{\boldsymbol{v}}_i) \tag{10.66}$$

Derivation. *Angular momentum of a system using the CoM*
Eq. (10.66) *can be simplified, as we will derive below, by utilising the properties of the CoM and these kinematic equations:*

$$\vec{\boldsymbol{r}}_{i/P} = \vec{\boldsymbol{r}}_{G/P} + \vec{\boldsymbol{r}}_{i/G} \tag{10.67}$$

$$\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_G + \vec{\boldsymbol{v}}_{i/G} \tag{10.68}$$

We substitute these two equations into Eq. (10.66), and use that the terms that do not contain the index i can be taken outside of the sums:

$$
\begin{aligned}
\vec{\boldsymbol{L}}_{C,P} &= \sum_i \vec{\boldsymbol{L}}_{i/P} = \sum_i (\vec{\boldsymbol{r}}_{G/P} + \vec{\boldsymbol{r}}_{i/G}) \times \left[m_i(\vec{\boldsymbol{v}}_G + \vec{\boldsymbol{v}}_{i/G})\right] && (10.69)\\
&= \vec{\boldsymbol{r}}_{G/P} \times \vec{\boldsymbol{v}}_G \left(\sum_i m_i\right) + \vec{\boldsymbol{r}}_{G/P} \times \left(\sum_i m_i \vec{\boldsymbol{v}}_{i/G}\right) \\
&+ \left(\sum_i m_i \vec{\boldsymbol{r}}_{i/G}\right) \times \vec{\boldsymbol{v}}_G + \sum_i m_i \vec{\boldsymbol{r}}_{i/G} \times \vec{\boldsymbol{v}}_{i/G} && (10.70)\\
&= \vec{\boldsymbol{r}}_{G/P} \times (m_{\text{tot}} \vec{\boldsymbol{v}}_G) + \sum_i m_i \vec{\boldsymbol{r}}_{i/G} \times \vec{\boldsymbol{v}}_{i/G} && (10.71)\\
&= \vec{\boldsymbol{L}}_{G/P} + \sum_i \vec{\boldsymbol{L}}_{i/G} && (10.72)
\end{aligned}
$$

We obtained four terms in Eq. (10.70), of which the middle two are zero, because it follows from the definition of the CoM Eq. (7.37) that if $P = G$ we have $\sum_i m_i \vec{\boldsymbol{r}}_{i/G} = (\sum_i m_i)\vec{\boldsymbol{r}}_{G/G} = \vec{\boldsymbol{0}}$, and the same holds for its time derivative $\sum_i m_i \vec{\boldsymbol{v}}_{i/G} = \vec{\boldsymbol{0}}$.

We have thus derived with Eq. (10.72) that the angular momentum of a system of point masses with respect to an arbitrary point P is identical to the sum of the angular momentum of a point mass equal to the total mass m_{tot} of the system at the CoM with respect to point P, and the angular momentum of the system with respect to point G, its CoM.

10.9.3 Angular momentum of a rigid body

The term $\sum_i \vec{\boldsymbol{L}}_{i/G}$ in the angular momentum expression Eq. (10.72) is still a bit difficult to evaluate, since it is a sum over many point masses. It can be simplified for rigid bodies by making use of the property that we derived in the previous chapter that all point masses in a rigid body C have the same angular velocity ω_C.

Derivation. *Angular momentum of a rigid body with respect to its CoM.*

From Eq. (9.26) we have the kinematic equation that holds for all point masses i in the rigid body:

$$\vec{\boldsymbol{v}}_{i/G} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{i/G} \tag{10.73}$$

We substitute this equation in Eq. (10.71) for the angular momentum $\vec{\boldsymbol{L}}_{C,G} = \sum_i \vec{\boldsymbol{L}}_{i/G}$ for a rigid body with respect to its CoM:

$$\sum_i \vec{\boldsymbol{L}}_{i/G} = \sum_i m_i \vec{\boldsymbol{r}}_{i/G} \times \vec{\boldsymbol{v}}_{i/G} \tag{10.74}$$

$$= \sum_i m_i \vec{\boldsymbol{r}}_{i/G} \times \left(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{i/G}\right) \tag{10.75}$$

$$= \sum_i m_i \left[(\vec{\boldsymbol{r}}_{i/G} \cdot \vec{\boldsymbol{r}}_{i/G})\vec{\boldsymbol{\omega}} - (\vec{\boldsymbol{r}}_{i/G} \cdot \vec{\boldsymbol{\omega}})\vec{\boldsymbol{r}}_{i/G}\right] \tag{10.76}$$

$$\sum_i \vec{\boldsymbol{L}}_{i/G,2D} = \sum_i m_i[(x_{i/G}^2 + y_{i/G}^2 + z_{i/G}^2)\omega\hat{\boldsymbol{k}}$$

$$-(z_{i/G}\omega)(x_{i/G}\hat{\boldsymbol{\imath}} + y_{i/G}\hat{\boldsymbol{\jmath}} + z_{i/G}\hat{\boldsymbol{k}})] \tag{10.77}$$

$$= \sum_i m_i \left[\rho_{i/G}^2\omega\hat{\boldsymbol{k}} - z_{i/G}\omega(x_{i/G}\hat{\boldsymbol{\imath}} + y_{i/G}\hat{\boldsymbol{\jmath}})\right] \tag{10.78}$$

$$= (I_{zz}\hat{\boldsymbol{k}} + I_{xz}\hat{\boldsymbol{\imath}} + I_{yz}\hat{\boldsymbol{\jmath}})\omega \tag{10.79}$$

In this derivation we used this vector identity for a triple product to obtain Eq. (10.76): $\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}})\vec{\boldsymbol{b}} - (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}})\vec{\boldsymbol{c}}$. In Eq. (10.77) we introduced a Cartesian coordinate system with origin G and z-axis in the direction $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$, such that $\vec{\boldsymbol{r}}_{i/G} = x_{i/G}\hat{\boldsymbol{\imath}} + y_{i/G}\hat{\boldsymbol{\jmath}} + z_{i/G}\hat{\boldsymbol{k}}$. In Eq. (10.78) we write the expression in terms of the distance $\rho_{i/G}$ to the z-axis with $\rho_{i/G}^2 = x_{i/G}^2 + y_{i/G}^2$. We also find that in 3D the angular momentum vector $\sum_i \vec{\boldsymbol{L}}_{i/G}$ is not necessarily parallel to the angular velocity vector $\vec{\boldsymbol{\omega}}$ since it also contains a term along the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions. In the last equation Eq. (10.79) we simplify the expression by introducing the components I_{zz}, I_{xz} and I_{yz} of the inertia tensor $\mathbf{I}_C$ of the rigid body C, that follow from the equation $\vec{\boldsymbol{L}} = \mathbf{I}\vec{\boldsymbol{\omega}}$. The two subscripts, e.g. in I_{zz}, indicate the relation between the L_z and ω_z vector component.

10.9.4 Derivation of the inertia tensor

Note that a similar equation to the parallel axis theorem can be derived for the products of inertia by substituting $\vec{\boldsymbol{r}}_{i/P} = \vec{\boldsymbol{r}}_{i/G} + \vec{\boldsymbol{r}}_{G/P}$ in Eq. (10.76) and Eq. (10.79), while using the properties of the CoM:

$$I_{P,xz} = -m_{\text{tot}}x_{G/P}z_{G/P} + I_{G,xz} \tag{10.80}$$

The full inertia tensor with respect to an arbitrary point P can thus be written as the sum of an inertia tensor of the CoM $\mathbf{I}_{G,P}$ and an inertia tensor $\mathbf{I}_{C,G}$ of object C with respect to the CoM:

$$\mathbf{I}_{C,P} = \mathbf{I}_{G,P} + \mathbf{I}_{C,G} \tag{10.81}$$

$$\mathbf{I}_{G,P} = m_{\text{tot}} \begin{bmatrix} (y_{G/P}^2 + z_{G/P}^2) & -x_{G/P}y_{G/P} & -x_{G/P}z_{G/P} \\ -y_{G/P}x_{G/P} & (x_{G/P}^2 + z_{G/P}^2) & -y_{G/P}z_{G/P} \\ -z_{G/P}x_{G/P} & -z_{G/P}y_{G/P} & (x_{G/P}^2 + y_{G/P}^2) \end{bmatrix} \tag{10.82}$$

$$\mathbf{I}_{C,G} = \begin{bmatrix} I_{G,xx} & I_{G,xy} & I_{G,xz} \\ I_{G,yx} & I_{G,yy} & I_{G,yz} \\ I_{G,zx} & I_{G,zy} & I_{G,zz} \end{bmatrix} \tag{10.83}$$

10.10 Derivation of Euler's second law*

After having introduced the equations for angular momentum and moment of inertia, we are now ready to analyse the effect of moments on the rotation of rigid bodies, and will show how Euler's second law that was introduced in Eq. (10.21), can be derived.

10.10.1 Euler's second law

We first extend Eq. (10.15) to a system of point masses C, with respect to the same reference point P. The resultant moment $\vec{\boldsymbol{M}}_{C/P}$ on C is then given by:

$$\vec{\boldsymbol{M}}_{C/P} = \sum_i \vec{\boldsymbol{M}}_{i/P} = \sum_i \vec{\boldsymbol{r}}_{i/P} \times (m_i \vec{\boldsymbol{a}}_i) \tag{10.84}$$

To see how the right term of this equation is related to the properties of the total angular momentum $\vec{\boldsymbol{L}}_{C/P}$ of C, we compare it to the time derivative of Eq. (10.66):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{L}}_{C/P} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \vec{\boldsymbol{L}}_{i/P} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \vec{\boldsymbol{r}}_{i/P} \times (m_i \vec{\boldsymbol{v}}_i) & (10.85) \\ &= \sum_i [m_i \vec{\boldsymbol{v}}_{i/P} \times \vec{\boldsymbol{v}}_i + \vec{\boldsymbol{r}}_{i/P} \times (m_i \vec{\boldsymbol{a}}_i)] & (10.86) \end{aligned}$$

To simplify this equation further, we would like to eliminate the terms $\vec{\boldsymbol{v}}_{i/P} \times \vec{\boldsymbol{v}}_i$. There are two ways to ensure this:

1. Use a fixed reference point P that has zero velocity and acceleration $\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{a}}_P = \vec{\boldsymbol{0}}$ with respect to the origin of an IRF. Then $\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_{i/P} + \vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{i/P}$.

Substitution into Eq. (10.85) results in a term $\vec{\boldsymbol{v}}_{i/P} \times \vec{\boldsymbol{v}}_{i/P} = \vec{\mathbf{0}}$. Thus for a fixed point P we obtain from Eq. (10.86):

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i \vec{\boldsymbol{L}}_{i/P} = \sum_i \vec{\boldsymbol{r}}_{i/P} \times (m_i \vec{\boldsymbol{a}}_i) \tag{10.87}$$

2. Choose the CoM as the reference point. Then we substitute $\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_{i/G} + \vec{\boldsymbol{v}}_G$ and get $\sum_i m_i \vec{\boldsymbol{v}}_{i/G} \times (\vec{\boldsymbol{v}}_{i/G} + \vec{\boldsymbol{v}}_G)$, where the terms $\vec{\boldsymbol{v}}_{i/G} \times \vec{\boldsymbol{v}}_{i/G} = \vec{\mathbf{0}}$ are zero and where the term $[\sum_i m_i(\vec{\boldsymbol{v}}_i - \vec{\boldsymbol{v}}_G)] \times \vec{\boldsymbol{v}}_G$ is also zero because $\sum_i m_i \vec{\boldsymbol{v}}_i = m_{\text{tot}} \vec{\boldsymbol{v}}_G$. Then we obtain from Eq. (10.86) and that equation is even valid if G has a nonzero velocity or acceleration:

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i \vec{\boldsymbol{L}}_{i/G} = \sum_i \vec{\boldsymbol{r}}_{i/G} \times (m_i \vec{\boldsymbol{a}}_i) \tag{10.88}$$

Since the right side of Eq. (10.84) is identical to that of Eq. (10.87) and Eq. (10.88), we obtain Euler's second law.

Concept. *Euler's second law*
Euler's second law states that the total external moment acting on C equals the time derivative of the total angular momentum of C, as long as a reference point P is chosen that is fixed in an IRF or the CoM G is used as reference point.

$$\vec{\boldsymbol{M}}_{C/P,\text{ext}} = \frac{\mathrm{d}}{\mathrm{d}t} \vec{\boldsymbol{L}}_{C/P} \tag{10.89}$$

$$\vec{\boldsymbol{M}}_{C/G,\text{ext}} = \frac{\mathrm{d}}{\mathrm{d}t} \vec{\boldsymbol{L}}_{C/G} \tag{10.90}$$

This equation relates moment and angular momentum in a similar way as Newton's second law $\vec{\boldsymbol{F}} = \frac{\mathrm{d}\vec{\boldsymbol{p}}}{\mathrm{d}t} = m\vec{\boldsymbol{a}}$ relates force and momentum. The subscript ext indicates that only external moments contribute to the resultant moment on the rigid body.

10.10.2 Euler's second law for a rigid body

Euler's second law can be applied to rigid bodies by using the properties of the angular momentum of rigid bodies that we have derived earlier in this chapter in Eq. (10.71) and the inertia tensor Eq. (10.30), which can be combined to obtain the total angular momentum of a rigid body C:

$$\vec{\boldsymbol{L}}_{C,P} = \vec{\boldsymbol{r}}_{G/P} \times (m_{\text{tot}} \vec{\boldsymbol{v}}_{G/P}) + \mathbf{I}_{C,G} \vec{\boldsymbol{\omega}}_C \tag{10.91}$$

This equation is substituted in Eq. (10.89). By taking the time derivative of Eq. (10.91), using that $\vec{a}_P = \vec{0}$ in an IRF and using that $\vec{v}_{G/P} \times \vec{v}_{G/P} = \vec{0}$ we obtain:

Concept. *Euler's second law for a rigid body C*

$$\sum_i \vec{M}_{C,i/P,\text{ext}} = \vec{r}_{G/P} \times (m_{\text{tot}}\vec{a}_G) + \frac{\text{d}}{\text{d}t}(\mathbf{I}_{C,G})\,\vec{\omega}_C + \mathbf{I}_{C,G}\vec{\alpha}_C \tag{10.92}$$

Eq. (10.92) shows that the resultant moment acting on a rigid body equals the sum of the of three components. We will discuss all three of them, in particular their relevance for the analysis of the planar kinetics of rigid bodies.

First there is the term $\vec{r}_{G/P} \times (m_{\text{tot}}\vec{a}_G)$, which is due to the angular momentum of the centre of mass $\vec{L}_{G/P}$ and becomes zero if the CoM is chosen as a reference point because $\vec{r}_{G/G} = \vec{0}$. Note that from Eq. (8.11) we have $\sum_i \vec{F}_{i,\text{ext}} = m_{\text{tot}}\vec{a}_G$, which can help to evaluate $\vec{a}_G$ and the contribution of this term. Note that if we are dealing with point masses, we only have this first term.

The second term is $\frac{\text{d}}{\text{d}t}\mathbf{I}_{C,G}\vec{\omega}_C$ is relevant when the inertia tensor changes in time. In the previous section we calculated $\mathbf{I}_{C,G}$ with respect to a Cartesian coordinate system. When an object changes orientation with respect to that system, its inertia tensor can change because the moment and product of inertia integrals change. However, in 2D planar kinetics all objects move in the xy-plane and all forces are tangential to the xy-plane, such that all angular velocity and angular acceleration acceleration vectors point in the z-direction, and the same holds for the moment vectors. In planar kinetics, which is the focus of this textbook, the orientation of the rigid body therefore does not change such that we always have that the time derivative of the inertia tensor $\frac{\text{d}}{\text{d}t}\mathbf{I}_{C,G}$ is zero, which substantially simplifies the dynamics, but eliminates the occurrence of special dynamic effects like precession that will be dealt with when treating 3D dynamics. Note that another way to ensure that the orientation of the rigid body with respect to the coordinate system does not change is to fix the coordinate system to the rigid body, and have it rotate along with it. This is often the method of choice for analysing more advanced rigid body dynamics problems but the drawback of that is that you are not in an IRF anymore.

Finally we have the term $\mathbf{I}_{C,G}\vec{\alpha}_C$. As discussed in the previous section, in planar kinetics we only deal with rigid bodies with diagonal inertia tensors, with both $\vec{M}_{C,P}$ and $\vec{\alpha}_C$ pointing along the z axis, such that I_{zz} is the only relevant term in the inertia tensor, such that this term becomes $I_{zz}\alpha_C\hat{\boldsymbol{k}}$. From these considerations, we can significantly simplify Euler's second law for the case of 2D motion in the xy-plane:

Concept. *Euler's second law for planar 2D kinetics of rigid body C*

$$\sum_i M_{C,i/P,2D}\hat{\boldsymbol{k}} = \vec{\boldsymbol{r}}_{G/P} \times (m_{\text{tot}}\vec{\boldsymbol{a}}_G) + I_{G,zz}\alpha_C\hat{\boldsymbol{k}} \tag{10.93}$$

This important equation is much easier to deal with than Eq. (10.92) and can be used to analyse many dynamic systems, as long as we keep in mind that it is only valid for 2D planar systems.

10.11 Summary

In this chapter we have introduced the concepts needed to analyse the planar kinetics of rigid bodies. The moments of forces were introduced, the angular momentum of rigid bodies was calculated. The moment of inertia was defined and methods to determine it for solid continuous bodies were introduced. Finally, this allowed us to determine Euler's second law, and the equation of motion for rotations of rigid bodies. In essence our toolbox to analyse the planar dynamics of rigid bodies and point masses is quite complete now. However, in some cases the principles of work and energy, and that of impulse and momentum, can facilitate the analysis of the kinetics of rigid bodies. We will extend these concepts and apply them to rotations in the next two chapters.

- **Euler's first law**

$$\sum \vec{\boldsymbol{F}}_{\text{ext}} = m_{\text{tot}}\vec{\boldsymbol{a}}_G \tag{10.94}$$

$$\sum F_{\text{ext},x} = m_{\text{tot}}\ddot{x}_G \tag{10.95}$$

$$\sum F_{\text{ext},y} = m_{\text{tot}}\ddot{y}_G \tag{10.96}$$

  - The sum of internal forces on a rigid body or system is always zero.

- **Resultant moment on a rigid body**

$$\vec{\boldsymbol{M}}_{C/P,\text{ext}} = \sum_i \vec{\boldsymbol{M}}_{i/P,\text{ext}} = \sum_i \vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_{i,\text{ext}} \tag{10.97}$$

  - The sum of internal moments on a rigid body or system is always zero.

- **Total angular momentum of a rigid body**

$$\vec{\boldsymbol{L}}_{C/P} = \mathbf{I}_{C/P}\vec{\boldsymbol{\omega}}_C \tag{10.98}$$

$$\vec{\boldsymbol{L}}_{C/P,2D} = \left(m_{\text{tot}}\rho^2_{G/P} + I_{C/G,zz}\right)\omega_C\hat{\boldsymbol{k}} \tag{10.99}$$

- **Moment of inertia I_{zz} of a rigid body**

$$I_G = \sum_i m_i\rho^2_{i/G} = \int_V \rho_m\rho^2_{i/G}\mathrm{d}V \tag{10.100}$$

  - Techniques for solving these kind of moment of inertia sums and integrals.
  - Parallel axis theorem:

$$I_P = m_{\text{tot}}\rho^2_{G/P} + I_G \tag{10.101}$$

  - Adding and subtracting moments of inertia:

$$I_{C,P} = I_{A,P} \pm I_{B,P} \tag{10.102}$$

- **Euler's second law**

$$\vec{\boldsymbol{M}}_{C/P} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{\boldsymbol{L}}_{C/P} \tag{10.103}$$

$$M_{C/P,2D}\hat{\boldsymbol{k}} = \vec{\boldsymbol{r}}_{G/P} \times (m_{\text{tot}}\vec{\boldsymbol{a}}_G) + I_{C/G,zz}\alpha_C\hat{\boldsymbol{k}} \tag{10.104}$$

  - Only valid if reference point P is a fixed point in an IRF, or if $P = G$.
  - Only valid if the same reference point P is used for determining $\vec{\boldsymbol{M}}_{C/P}$ and $\vec{\boldsymbol{L}}_{C/P}$.

- **Determine and solve the EoM for a rigid body**
  - Methodology to determine the EoM, using FBD, moments, constraints and Euler's laws.
  - Choose fixed axis as reference point when dealing with pure rotation around fixed point.
  - Simplify EoMs
  - Solve EoMs using kinematic techniques.

- **Couples**
  - A *couple* is a number of force vectors that generate a moment on a system for which the sum of forces is zero such that $\vec{\boldsymbol{a}}_G = \vec{\mathbf{0}}$. For a *simple couple* the number of forces is two.
  - The resultant moment vector of a couple is independent of the choice of reference point P. Such a vector is called a *pure moment* or *couple moment.*

11 Work and Energy of Rigid Bodies

In the previous two chapters we have outlined the methodology for analysing the kinematics and kinetics of rigid bodies. However, solving their equations of motion by integration or other means is not always necessary. In this chapter we introduce energy methods to analyse their dynamics, similar to what we did for point masses in Ch. 7 but now also including moments and rotational motion. In the first sections we will derive the concepts of work and kinetic energy for rigid bodies.

11.1 Work on a rigid body

11.1.1 Work of a force on a rigid body

If an external force $\vec{\boldsymbol{F}}_B$ acts on a rigid body on a single point of action, then the work it does is equal to the work it would do on a point mass that is located on the point of action $\vec{\boldsymbol{r}}_B$. So, we can just use Eq. (7.2) to evaluate the work done by such a force.

Concept. *Work of a force on a rigid body*

$$W_{B,1\to 2} = \int_{\vec{\boldsymbol{r}}_1}^{\vec{\boldsymbol{r}}_2} \vec{\boldsymbol{F}}_{B,\text{ext}} \cdot \mathrm{d}\vec{\boldsymbol{r}}_B \tag{11.1}$$

It is important to note that because a rigid body is not deformable, the work done by internal forces on the rigid body is zero. This is obvious when evaluating the work done in a reference frame where the rigid body is at rest, since there all displacements $\mathrm{d}\vec{\boldsymbol{r}}$ are zero, and is still valid when the rigid body moves as a consequence of Newton's third law. Since it is not deformable, a rigid body also cannot store potential energy. Forces from deformable structures, like springs, can perform work, even if they are internal components in a system. So, if you have a system consisting of multiple elements, e.g. two rigid bodies connected by a spring, then you do have to consider the internal forces of that system when determining the work done on the system.

11.1.2 Work of a couple on a rigid body

Concept. *Work of a couple*

The work done by a pure moment or couple on a rigid body is given by the integral over the dot product between the moment vector and the infinitesimal angular displacement vector $\mathrm{d}\vec{\boldsymbol{\phi}} = \mathrm{d}\phi\hat{\boldsymbol{k}}$.

$$W_{\text{couple},1\to2} = \int_{\phi_1}^{\phi_2} \vec{\boldsymbol{M}}_{\text{couple}} \cdot \mathrm{d}\vec{\boldsymbol{\phi}} \tag{11.2}$$

Derivation. *Work of a couple*

A simple couple $\vec{\boldsymbol{M}}_{\text{couple}}$ is generated by 2 equal and opposite external forces $\vec{\boldsymbol{F}}_A = -\vec{\boldsymbol{F}}_B$ that act on points A and B on the rigid body. We derive the work done by the couple when the rigid body purely rotates around fixed point A (e.g. because the point is chosen to be at the CoM: $A = G$) such that the work done by $\vec{\boldsymbol{F}}_A$ is zero and we only need to consider the work done by $\vec{\boldsymbol{F}}_B$.

$$\begin{aligned}
W_{\text{couple},1\to2} &= \int_{\vec{\boldsymbol{r}}_1}^{\vec{\boldsymbol{r}}_2} \vec{\boldsymbol{F}}_B \cdot \mathrm{d}\vec{\boldsymbol{r}}_B && (11.3)\\
&= \int_{t_1}^{t_2} \vec{\boldsymbol{F}}_B \cdot \vec{\boldsymbol{v}}_B \mathrm{d}t && (11.4)\\
&= \int_{t_1}^{t_2} \vec{\boldsymbol{F}}_B \cdot (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A}) \mathrm{d}t && (11.5)\\
&= \int_{t_1}^{t_2} (\vec{\boldsymbol{r}}_{B/A} \times \vec{\boldsymbol{F}}_B) \cdot \vec{\boldsymbol{\omega}} \mathrm{d}t && (11.6)\\
&= \int_{\phi_1}^{\phi_2} \vec{\boldsymbol{M}}_{\text{couple}} \cdot \mathrm{d}\vec{\boldsymbol{\phi}} && (11.7)
\end{aligned}$$

Here we used the vector identity $\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = (\vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}) \cdot \vec{\boldsymbol{b}}$ and used Eq. (10.53). Note that the moment of the couple, and the resulting expression for the work of a couple are independent of the reference point for rotations.

11.1.3 Work of a force field

The situation becomes more complicated when the rigid body moves in a force field. Then the total work of the force field can be determined by integrating or summing the work over all point masses in the rigid body. This can sometimes be quite complex if the force field changes with position. But it is easier for a constant gravitational force field which we already analysed in Eq. (7.43), where we found that:

$$W_{g,1\to 2} = -\Delta V_g = -m_{\text{tot}}\Delta\vec{\boldsymbol{r}}_G \cdot \hat{\boldsymbol{\jmath}} = -m_{\text{tot}}\Delta h_G \tag{11.8}$$

So the work done on the whole rigid body by the gravitational force is equal to the height reduction of the CoM times the total mass of the rigid body. Note that the moment of the gravitational force on a rigid body is zero when the CoM is taken as reference point, therefore it does no work when the rigid body rotates around its CoM. This can be derived as follows.

Derivation. *The moment of gravity on a system or rigid body is zero with respect to its CoM.*

$$\vec{\boldsymbol{M}}_{g/G} = \sum_i \vec{\boldsymbol{r}}_{i/G} \times \vec{\boldsymbol{F}}_{i,g} = \sum_i \vec{\boldsymbol{r}}_{i/G} \times (-m_i g\hat{\boldsymbol{\jmath}}) \tag{11.9}$$

$$= \left(\sum_i m_i(\vec{\boldsymbol{r}}_i - \vec{\boldsymbol{r}}_G)\right) \times (-g\hat{\boldsymbol{\jmath}}) \tag{11.10}$$

$$= [m_{\text{tot}}(\vec{\boldsymbol{r}}_G - \vec{\boldsymbol{r}}_G)] \times (-g\hat{\boldsymbol{\jmath}}) = \vec{\boldsymbol{0}}, \tag{11.11}$$

where we used the property of the CoM that $m_{\text{tot}}\vec{\boldsymbol{r}}_G = \sum_i m_i\vec{\boldsymbol{r}}_i$.

11.2 Kinetic energy of a rigid body

As will be derived in Sec. 11.6, the kinetic energy of a rigid body C can be written as a sum of a translational component $T_{C,\text{trans}}$ due to the speed $\vec{\boldsymbol{v}}_G$ of the CoM and a rotational component $T_{C,\text{rot}}$ due to its angular velocity ω_C.

Concept. *Kinetic energy of a rigid body C*

$$T_{C,2D} = T_{C,\text{trans}} + T_{C,\text{rot},2D} \tag{11.12}$$

$$T_{C,\text{trans}} = \frac{1}{2}m_{\text{tot}}|\vec{\boldsymbol{v}}_G|^2 \tag{11.13}$$

$$T_{C,\text{rot},2D} = \frac{1}{2}I_{G,zz}\omega_C^2 \tag{11.14}$$

11.3 Principle of work and energy

The principle of work and energy is the same for rigid bodies and rotations as for point masses discussed in Ch. 7. The main difference when working with rigid bodies is that we include the work of moments and the kinetic energy due to rotation of the rigid body. A second important point to consider is that internal forces inside the rigid body do not perform work. So, we sum over all work done by forces and all kinetic energies to obtain:

Concept. *Principle of work and energy for rigid bodies*

$$W_{\text{tot}} = \sum W_{1\to 2} = \sum T_2 - \sum T_1 = \Delta T_{\text{tot}} \tag{11.15}$$

11.4 Conservation of energy and potential energy

Like in Sec. 7.2, the analysis of rigid bodies using work and energy principles is facilitated when only conservative forces are acting on the system. For this situation, the work done can be described in terms of the change in potential energy since:

$$W_{\text{tot,cons}} = -\Delta V_{\text{tot}} \tag{11.16}$$

Combining this with Eq. (11.15) we obtain, just like in Eq. (7.13) that the total energy does not change:

Concept. *Law of energy conservation for rigid bodies*

$$\Delta V_{\text{tot}} + \Delta T_{\text{tot}} = 0 \tag{11.17}$$

And for a large number of objects, the conservation of energy equation between two times t_1 and t_2 is like Eq. (7.34):

$$\sum_i T_i(t_1) + \sum_i V_i(t_1) = \sum_i T_i(t_2) + \sum_i V_i(t_2) \tag{11.18}$$

We can still use the potential energy expression of forces derived in Sec. 7.3.

S **Example 11.1** *Determine the speed of the yo-yo with radius R in Fig. 11.1 after falling a distance Δh from rest.*
We first note that the kinetic T_1 and potential energy V_1 are zero at $t = t_1$, since the yo-yo starts from rest. Then at $t = t_2$ its gravitational potential energy has reduced to a value $V_2 = -mg\Delta h$. Since the force of the rope and the gravitational force are conservative forces, we can apply the law of energy conservation to determine the kinetic energy of the yo-yo at t_2:

$$\begin{aligned} T_1 + V_1 &= T_2 + V_2 && (11.19)\\ 0 + 0 &= T_2 - mg\Delta h && (11.20)\\ T_2 &= mg\Delta h && (11.21) \end{aligned}$$

To solve the problem we need the constraint equation set by the rope. If the angle ϕ increases, the rope releases from the yo-yo and the y-coordinate of the CoM of the yo-yo decreases:

$$\begin{aligned} -y_G &= \Delta h = \phi R + L_0 && (11.22)\\ v_{G,y} &= \dot{y}_G = -\omega R, && (11.23) \end{aligned}$$

where we took the time derivative of the constraint equation set by the rope of the yo-yo to relate it CoM velocity to its angular velocity. Now we write the equation for the kinetic energy of the yo-yo T_2, which is the sum of the translational and rotational kinetic energy:

$$\begin{aligned} T_2 &= \frac{1}{2}m|\vec{\boldsymbol{v}}_G|^2 + \frac{1}{2}I_G\omega^2 && (11.24)\\ &= \frac{1}{2}mv_{G,y}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)(v_{G,y}/R)^2 && (11.25)\\ &= \frac{3}{4}mv_{G,y}^2 = mg\Delta h && (11.26)\\ |v_{G,y}| &= \sqrt{\frac{4}{3}g\Delta h}, && (11.27) \end{aligned}$$

where we used that for a solid disk $I_G = \frac{1}{2}mR^2$. Thus we obtain the velocity of the CoM of the yo-yo. Note that the energy method does not give us the direction of the velocity. But in this case, no forces act in the x direction and it is straightforward to determine that it is downward $\vec{\boldsymbol{v}}_G = -\sqrt{\frac{4}{3}g\Delta h}\hat{\boldsymbol{\jmath}}$. Note that this velocity is lower than that of a free falling disk because part of the gravitational work is converted to rotational kinetic energy.

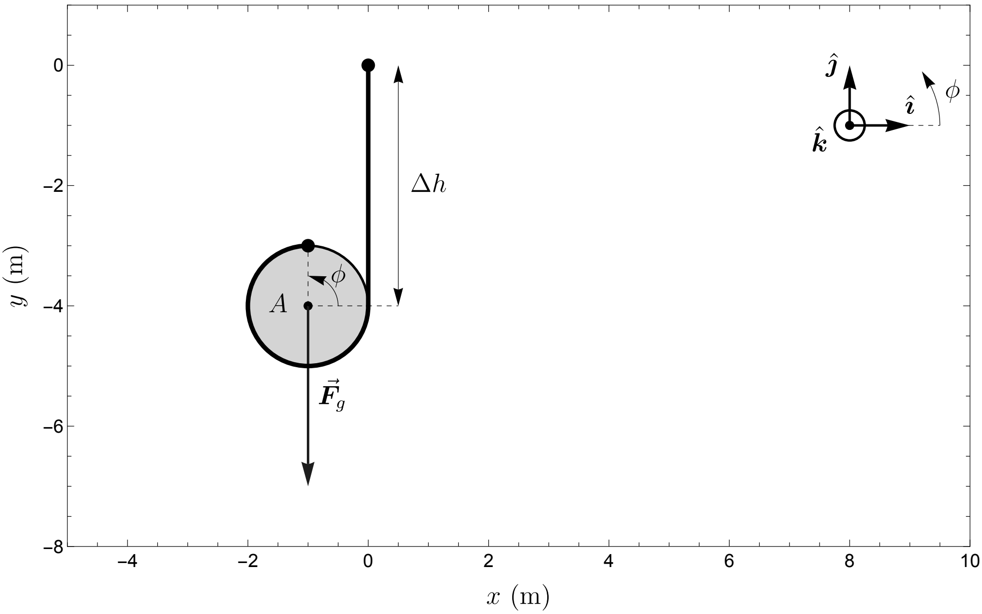


Figure 11.1: A yo-yo A is dropped from rest. Determine its velocity $\vec{v}_G$ after falling a distance Δh.

11.5 Potential energy of a torsion spring

A special type of couple is a torsion spring which generates a couple proportional to the angle by which it is rotated. For rotation of the torsion spring by an angle ϕ with respect to its zero angle at $\phi = 0$ around the z-axis, its angle-dependent moment is:

$$\vec{M}_{\text{couple}} = -\kappa\phi\hat{k}, \tag{11.28}$$

where κ is the torsion coefficient, having units of N·m/rad. Substituting this into Eq. (11.2) we obtain:

$$\Delta V_{\text{couple}} = \int_{\phi_{F,1}}^{\phi_{F,2}} \kappa\phi\hat{k} \cdot \mathrm{d}\phi\hat{k} \tag{11.29}$$

$$V_{\text{couple}} = \frac{1}{2}\kappa\phi^2 \tag{11.30}$$

11.6 Derivation kinetic energy of rigid body*

Derivation. *We now derive the kinetic energy T_C of a rigid body C, by summing the kinetic energies of all point masses i inside the rigid body.*

$$\begin{aligned}
T_C &= \sum_i \frac{1}{2} m_i |\vec{\boldsymbol{v}}_i|^2 && (11.31)\\
&= \frac{1}{2}\sum_i m_i(\vec{\boldsymbol{v}}_G + \vec{\boldsymbol{\omega}}_C \times \vec{\boldsymbol{r}}_{i/G}) \cdot (\vec{\boldsymbol{v}}_G + \vec{\boldsymbol{\omega}}_C \times \vec{\boldsymbol{r}}_{i/G}) && (11.32)\\
&= \frac{1}{2}\sum_i m_i |\vec{\boldsymbol{v}}_G|^2 + \frac{1}{2}\sum_i m_i |\vec{\boldsymbol{\omega}}_C \times \vec{\boldsymbol{r}}_{i/G}|^2 \\
&\quad + \frac{1}{2}\sum_i 2 m_i \vec{\boldsymbol{v}}_G \cdot (\vec{\boldsymbol{\omega}}_C \times \vec{\boldsymbol{r}}_{i/G}) && (11.33)\\
&= \frac{1}{2}\sum_i m_i |\vec{\boldsymbol{v}}_G|^2 + \frac{1}{2}\sum_i m_i |\vec{\boldsymbol{\omega}}_C \times \vec{\boldsymbol{r}}_{i/G}|^2 \\
&\quad + \left(\underbrace{\sum_i m_i \vec{\boldsymbol{r}}_{i/G}}_{=0}\right) \cdot (\vec{\boldsymbol{v}}_G \times \vec{\boldsymbol{\omega}}_C) && (11.34)\\
&= T_{C,\text{trans}} + T_{C,\text{rot}} && (11.35)\\
T_{C,\text{trans}} &= \frac{1}{2} m_{\text{tot}} |\vec{\boldsymbol{v}}_G|^2 && (11.36)\\
T_{C,\text{rot}} &= \frac{1}{2}\sum_i m_i |\vec{\boldsymbol{\omega}}_C \times \vec{\boldsymbol{r}}_{i/G}|^2, && (11.37)
\end{aligned}$$

where the kinematic equation $\vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_G + \vec{\boldsymbol{\omega}}_C \times \vec{\boldsymbol{r}}_{i/G}$ and the vector identity $\vec{\boldsymbol{a}} \cdot (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \vec{\boldsymbol{c}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}})$ were used. For 2D planar motion we obtain:

$$\begin{aligned}
T_{C,\text{rot,2D}} &= \frac{1}{2}\sum_i m_i |\omega_C \hat{\boldsymbol{k}} \times \rho_{i/G} \hat{\boldsymbol{\rho}}_G|^2 && (11.38)\\
&= \frac{1}{2}\left(\underbrace{\sum_i m_i \rho_{i/G}^2}_{=I_{G,zz}}\right)\omega_C^2 && (11.39)\\
&= \frac{1}{2} I_{G,zz}\omega_C^2 && (11.40)
\end{aligned}$$

In summary:

$$T_{C,2D} = T_{C,\text{trans}} + T_{C,\text{rot},2D} = \frac{1}{2} m_{\text{tot}} |\vec{\boldsymbol{v}}_G|^2 + \frac{1}{2} I_{G,zz}\omega_C^2 \qquad (11.41)$$

11.7 Summary

In this chapter we have extended the energy methods from Ch. 7 to rigid bodies. It was derived how moments contribute to the work on the rigid body and how the angular velocity of rigid bodies increases their kinetic energy. By including these effects, the principle of work and energy and the energy conservation concepts can also be applied to systems that include rigid bodies.

- **Work of an external force on a rigid body**

$$W_{B,1\to 2} = \int_{\vec{\boldsymbol{r}}_1}^{\vec{\boldsymbol{r}}_2} \vec{\boldsymbol{F}}_{B,\text{ext}} \cdot \mathrm{d}\vec{\boldsymbol{r}} \tag{11.42}$$

  - Internal forces in a rigid body do not perform work.

- **Work of a couple on a rigid body**

$$W_{\text{couple},1\to 2} = \int_{\phi_1}^{\phi_2} \vec{\boldsymbol{M}}_{\text{couple}} \cdot \mathrm{d}\vec{\boldsymbol{\phi}} \tag{11.43}$$

- **Potential energy of a couple**

$$V_{\text{couple}} = \frac{1}{2}\kappa\phi^2 \tag{11.44}$$

- **Kinetic energy of a rigid body**

$$T_{C,2D} = T_{C,\text{trans}} + T_{C,\text{rot},2D} \tag{11.45}$$

$$T_{C,\text{trans}} = \frac{1}{2}m_{\text{tot}}|\vec{\boldsymbol{v}}_G|^2 \tag{11.46}$$

$$T_{C,\text{rot},2D} = \frac{1}{2}I_{G,zz}\omega_C^2 \tag{11.47}$$

- **Principle of work and energy**

$$W_{\text{tot}} = \Delta T_{\text{tot}} \tag{11.48}$$

- **Conservation of energy when only conservative forces act**

$$\Delta V_{\text{tot}} + \Delta T_{\text{tot}} = 0 \tag{11.49}$$

12 Angular Impulse and Angular Momentum

In Ch. 8 we have introduced the principle of impulse and momentum and the law of momentum conservation that hold for systems of point masses and for rigid bodies. However, these laws do not provide information on the rotations, or changes in the angular velocity of rigid bodies. In this chapter we will extend these concepts to rotation of rigid bodies, thus providing a method to analyse both the changes in velocity of the CoM and the changes in the angular velocity $\vec{\boldsymbol{\omega}}$, when external forces and moments act on rigid bodies.

12.1 Principle of angular impulse and angular momentum

Derivation.

To derive the principle of angular momentum and angular impulse, we start from Euler's second law that was found in Ch. 10, Eq. (10.89):

$$\sum_i \vec{\boldsymbol{M}}_{i/P,\mathrm{ext}} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_i \vec{\boldsymbol{L}}_{i/P} \tag{12.1}$$

This equation holds for the angular momentum of a system of point masses with respect to a reference point P that is fixed in space in an inertial reference frame. The equation can be rewritten by multiplying both sides by $\mathrm{d}t$ and integrating over a time interval from t_1 to t_2:

$$\int_{t_1}^{t_2} \sum_i \vec{\boldsymbol{M}}_{i/P,\mathrm{ext}} \mathrm{d}t = \int_{t_1}^{t_2} \mathrm{d} \sum_i \vec{\boldsymbol{L}}_{i/P} = \sum_i \vec{\boldsymbol{L}}_{i/P}(t_2) - \sum_i \vec{\boldsymbol{L}}_{i/P}(t_1) \tag{12.2}$$

The integral on the left of Eq. (12.2) is defined as the total angular impulse acting on the system $\vec{\boldsymbol{H}}_{\mathrm{ang},P,12}$:

Concept. *Angular impulse*

$$\vec{\boldsymbol{H}}_{\text{ang},P,12} \equiv \int_{t_1}^{t_2} \sum_i \vec{\boldsymbol{M}}_{i/P,\text{ext}} \mathrm{d}t \tag{12.3}$$

Note that only external moments contribute to the angular impulse, since internal moments sum to zero just like in Euler's second law.

By rewriting Eq. (12.2) we obtain the principle of angular impulse and angular momentum, which states that the change in the angular momentum of the system is equal to the impulse generated by external moments on the system:

Concept. *Principle of angular impulse and angular momentum*
The change in angular momentum is equal to the angular impulse by external moments acting on the system.

$$\sum_i \vec{\boldsymbol{L}}_{i/P}(t_1) + \vec{\boldsymbol{H}}_{\text{ang},P,12} = \sum_i \vec{\boldsymbol{L}}_{i/P}(t_2) \tag{12.4}$$

Since we are mainly concerned with the analysis of kinetics of rigid bodies, we use Eq. (10.91) to derive the principle of angular impulse and angular momentum of a rigid body for the case of 2D planar kinetics:

Concept. *Principle of angular impulse and angular momentum for the planar 2D kinetics of a rigid body C*

$$\begin{aligned} \vec{\boldsymbol{r}}_{G/P,1} \times (m_{\text{tot}} \vec{\boldsymbol{v}}_{G,1}) + I_G \omega_1 \hat{\boldsymbol{k}} + \int_{t_1}^{t_2} M_{C/P} \hat{\boldsymbol{k}} \mathrm{d}t \\ = \vec{\boldsymbol{r}}_{G/P,2} \times (m_{\text{tot}} \vec{\boldsymbol{v}}_{G,2}) + I_G \omega_2 \hat{\boldsymbol{k}} \end{aligned} \tag{12.5}$$

If we choose $P = G$ this principle simplifies to:

$$I_G \omega_1 \hat{\boldsymbol{k}} + \int_{t_1}^{t_2} M_{C/G} \hat{\boldsymbol{k}} \mathrm{d}t = I_G \omega_2 \hat{\boldsymbol{k}} \tag{12.6}$$

12.2 Choosing a convenient reference point

Although the principle of angular impulse and angular momentum equation Eq. (12.5) can be used for any reference point P it can often be significantly simplified by a smart choice of the reference point P. Let us discuss some convenient choices:

1. The centre of mass G of the system of point masses. This causes all the cross products in Eq. (12.5) to disappear because $\vec{\boldsymbol{r}}_{G/G} = \vec{\mathbf{0}}$ and results in Eq. (12.6). Moreover, this is the only option for choosing an accelerating reference point.
2. Since the moment of an external force is $\vec{\boldsymbol{M}}_{i/P} = \vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_i$, choosing the point of action of that force as a reference point $P = i$ will result in the angular impulse of that point becoming zero because $\vec{\boldsymbol{r}}_{i/i} = \vec{\mathbf{0}}$. This is especially useful if the actual force $\vec{\boldsymbol{F}}_i$ is unknown.
3. In some cases the reference point P can be chosen such that either $\vec{\boldsymbol{v}}_G$ is parallel to $\vec{\boldsymbol{r}}_{G/P}$, or that one or more of the external forces $\vec{\boldsymbol{F}}_i$ is parallel to $\vec{\boldsymbol{r}}_{i/P}$, which can simplify the situation because either $\vec{\boldsymbol{r}}_{G/P} \times \vec{\boldsymbol{v}}_G = \vec{\mathbf{0}}$ or $\vec{\boldsymbol{r}}_{i/P} \times \vec{\boldsymbol{F}}_i = \vec{\mathbf{0}}$.

12.3 Conservation of angular momentum

A special situation is the case where the external angular impulse $\vec{\boldsymbol{H}}_{\text{ang},P}$ is zero. We substitute this condition in Eq. (12.4) and obtain that angular momentum does not change in this situation:

Concept. *Law of conservation of angular momentum*
If the total angular impulse generated by external moments on a system with respect to a fixed reference point P is zero ($\vec{\boldsymbol{H}}_{\text{ang},P} = \vec{\mathbf{0}}$), then the total angular momentum of the system is conserved, and does not change in time.

$$\sum_i \vec{\boldsymbol{L}}_{i/P}(t_1) = \sum_i \vec{\boldsymbol{L}}_{i/P}(t_2) \tag{12.7}$$

Whether or not conservation of angular momentum occurs can depend on the choice of the reference point P. For example when a single external force $\vec{\boldsymbol{F}}_i$ acts at a fixed point of action i on a system, the reference point P can be chosen at point i, or at any other point on the line of action of force $\vec{\boldsymbol{F}}_i$, to ensure that $\vec{\boldsymbol{H}}_{\text{ang},P} = \vec{\mathbf{0}}$ and the law of conservation of angular momentum holds.

An important case for conservation of angular momentum are systems on which there are no external moments and forces acting because in those systems the law of conservation of angular momentum holds for any choice of reference point P.

It is important to note that when analysing systems, the boundaries of the system can be chosen arbitrarily. By expanding the system boundaries to include all objects that generate forces, all forces will become internal forces, and conservation of (angular) momentum for this expanded system will hold. In the extreme case, the momentum and angular momentum of our whole universe are expected to be constant, since there are no external forces and moments acting on it, such that the law of conservation of momentum and angular momentum hold.

12.4 Using angular impulse and angular momentum

In contrast to a point mass, a rigid body has both translational and rotational degrees of freedom. By combining the methods from this chapter with those from Ch. 8, we can fully analyse the changes in both the velocities and angular velocities of rigid bodies. There are however two important limitations to these methods:

1. Changes in position are not determined by this method, so the motion needs to be known in advance, or the impulse needs to be so short that the change in position can be neglected.
2. Sufficient information about the forces and their points of action is needed to evaluate the angular impulse $\vec{\boldsymbol{H}}_{\text{ang},P,12}$. This is substantially easier if the position of the point of action of the forces is approximately constant (e.g. in short collisions) or if only pure couples are acting for which the point of action is irrelevant.

Finally we note that in cases where (angular) impulse and momentum do not provide enough information, it can sometimes be combined with energy methods to solve a problem, e.g. by combining conservation of (angular) momentum with conservation of energy. Let us discuss an example.

S **Example 12.1** Fig. 12.1 shows a ball A (a point mass), with mass m that collides elastically with a dumbbell at rest BC that consists of two point masses B and C with identical mass m.
Determine the velocities and angular velocities of the objects after the collision.
To solve this problem we first note that because the collision is elastic, all forces are conservative forces. Also we note that there are no external moments or forces acting on the system (gravity is not indicated and can thus be neglected).

Secondly, we need to choose a reference point for analysing the rotations of the system. A logical choice is to use the point $\vec{\boldsymbol{r}}_B$ as reference point. Since there are no external forces or moments acting on the system we have conservation of momentum and angular momentum, also we have conservation of energy. All forces and velocities are in the x direction, so we only need to consider their x components. This gives 3 equations, with three scalar unknowns after the collision: the velocity of mass A $v_{A,2,x}$, the velocity of the center of gravity of dumbbell BC $v_{G,2,x}$ and its

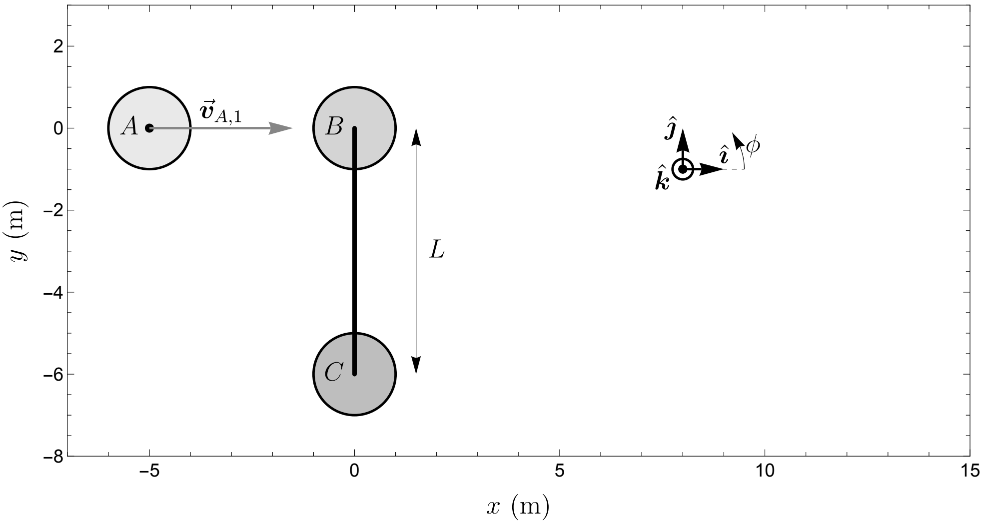


Figure 12.1: A ball collides with a dumbbell, determine the speeds and angular velocity of the dumbbell just after the collision.

angular velocity ω_2.

$$mv_{A,1,x} = \underbrace{mv_{A,2,x} + 2mv_{G,2,x}}_{\text{Conservation of momentum}} \tag{12.8}$$

$$\vec{r}_{B/B} \times m\vec{v}_{A,1} = \underbrace{\vec{r}_{B/B} \times m\vec{v}_{A,2} + \vec{r}_{G/B} \times (m_{BC}\vec{v}_{G,2}) + I_G\omega_2\hat{k}}_{\text{Conservation of angular momentum}}$$

$$\vec{0} = \vec{0} - L/2\hat{\jmath} \times (2mv_{G,2,x}\hat{\imath}) + I_G\omega_2\hat{k}$$

$$0 = mLv_{G,2,x} + I_G\omega_2 \tag{12.9}$$

$$\tfrac{1}{2}mv_{A,1,x}^2 = \underbrace{\tfrac{1}{2}mv_{A,2,x}^2 + mv_{G,2,x}^2 + \tfrac{1}{2}I_G\omega_2^2}_{\text{Conservation of energy}} \tag{12.10}$$

$$mv_{A,1,x}^2 = m(v_{A,1,x} - 2v_{G,2,x})^2 + 2mv_{G,2,x}^2 + (mLv_{G,2,x})^2/(\tfrac{1}{2}mL^2)$$

$$0 = 4v_{G,2,x}^2 - 4v_{A,1,x}v_{G,2,x} + 2v_{G,2,x}^2 + 2v_{G,2,x}^2$$

$$v_{G,2,x} = \tfrac{1}{2}v_{A,1,x} \tag{12.11}$$

$$v_{A,2,x} = \underbrace{0}_{\text{Using Eq. (12.8)}} \tag{12.12}$$

$$\omega_2 = \underbrace{(-mL/I_G)\tfrac{1}{2}v_{A,1,x} = -v_{A,1,x}/L}_{\text{Using Eq. (12.9)}} \tag{12.13}$$

Here we simultaneously solved the equations of momentum conservation, angular momentum conservation and energy conservation, while using that for a dumbbell, $I_G = \sum_i m_i\rho_{i/G}^2 = 2m(L/2)^2 = \frac{1}{2}mL^2$.

12.5 Summary

In this section we have extended the principles of momentum and impulse to rigid bodies. This involved introducing the angular impulse and angular momentum, and the principle relating them. In addition we discussed the law of conservation of angular momentum, which holds when the total external angular impulse on a system is zero.

- Angular impulse

$$\vec{\boldsymbol{H}}_{\mathrm{ang},P,12} \equiv \int_{t_1}^{t_2} \sum_i \vec{\boldsymbol{M}}_{i/P,\mathrm{ext}} \mathrm{d}t \tag{12.14}$$

- Principle of angular impulse and angular momentum

$$\sum_i \vec{\boldsymbol{L}}_{i/P}(t_1) + \vec{\boldsymbol{H}}_{\mathrm{ang},P,12} = \sum_i \vec{\boldsymbol{L}}_{i/P}(t_2) \tag{12.15}$$

$$\text{In 2D}: \quad \vec{\boldsymbol{r}}_{G/P,1} \times (m_{\mathrm{tot}} \vec{\boldsymbol{v}}_{G,1}) + I_G \omega_1 \hat{\boldsymbol{k}} + \int_{t_1}^{t_2} M_{C/P} \hat{\boldsymbol{k}} \mathrm{d}t = \vec{\boldsymbol{r}}_{G/P,2} \times (m_{\mathrm{tot}} \vec{\boldsymbol{v}}_{G,2}) + I_G \omega_2 \hat{\boldsymbol{k}} \tag{12.16}$$

$$I_G \omega_1 \hat{\boldsymbol{k}} + \int_{t_1}^{t_2} M_{C/G} \hat{\boldsymbol{k}} \mathrm{d}t = I_G \omega_2 \hat{\boldsymbol{k}} \tag{12.17}$$

- Conservation of angular momentum if the angular impulse is zero:

$$\sum_i \vec{\boldsymbol{L}}_{i/P}(t_1) = \sum_i \vec{\boldsymbol{L}}_{i/P}(t_2) \tag{12.18}$$

Part IV

Vibrations and Strategy

13 Vibrations

13.1 Introduction

In our daily life there are many types of periodic phenomena. The day-night rhythm, the waves on the sea, the trees swaying in the wind, a child going back and forth on a swing and the beating of our heart. Any type of periodic motion is called an oscillation. If an oscillation is of a mechanical nature, it is called a vibration.

Vibrations can be very useful, for example in a pendulum or quartz clock to keep track of time, or for generating sound tones with a speaker or musical instrument. Vibrations can also be very detrimental in the case of shaking buildings excited by earthquakes or in the uncontrolled resonant amplitude increase (flutter) of aircraft wings. It is therefore important for engineers to have good models for vibrations, both to use them for engineering vibrating devices, and for engineering methods to prevent their detrimental effects.

In this chapter we will discuss several types of vibrations that are a direct consequence of the combination of Newton's second law and the properties of a position or velocity dependent restoring force like that of a spring or a damper. When writing down the equation of motion for such a system, according to the methods discussed in Ch. 6, a second-order differential equation arises that we have not yet encountered before. It should be noted that analysis of vibrations can be performed using the methods of kinetics that we have already dealt with before, the main challenge in this chapter is solving the special types of differential equations that arise in vibration problems.

13.2 Free undamped vibrations

A mass-spring system is one of the simplest systems in which vibrations can occur. It consists of a point mass A, at position coordinate $y_{A,\mathrm{tot}}$ that is hanging from the ceiling by a spring with spring constant k. Such a system is a type of *mechanical resonator*. The motion of such a system is called *free undamped vibration*. It is *free* or undriven because there is no external periodic or time-dependent driving force, and it is *undamped* because there is

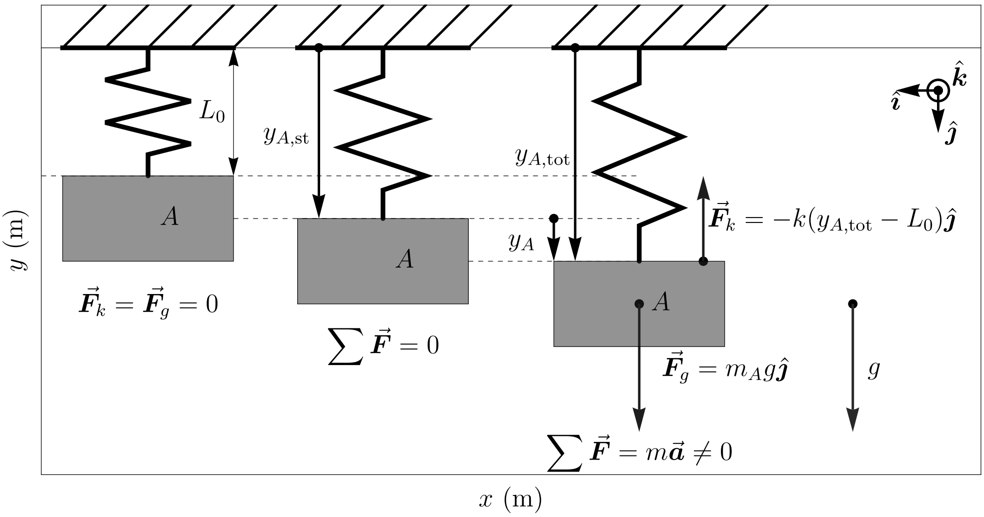


Figure 13.1: A mass A is suspended by a spring. When the spring is relaxed it has a length L_0. When it is in static equilibrium with the gravitational force, the spring has a length $L_0 + y_{A,\mathrm{st}}$. The displacement of the mass with respect to this static equilibrium position is measured by position coordinate $y_A(t)$.

no damping force acting on the mass.

Fig. 13.1 shows a sketch of what happens if a mass is suspended by a spring. First of all the spring has a relaxed length L_0 even if there are no forces acting on it. Then if the mass A is released it moves down by a distance $y_{A,\mathrm{st}}$ to a static equilibrium position for which the sum of forces is zero. Finally, if the mass is displaced from that position by an amount y_A the sum of forces is not zero such that the mass accelerates.

We determine the free vibrating motion of a mechanical resonator similarly to other kinetic problems, using the methods of Ch. 6. First we determine the EoMs and then we solve them. The sketch and coordinate system are in Fig. 13.1 and we draw the FBD in Fig. 13.2 to obtain the EoM:

$$\sum F_y = m_A g - k(y_{A,\mathrm{tot}} - L_0) = m_A \ddot{y}_{A,\mathrm{tot}} \tag{13.1}$$

This EoM is a so-called *second-order ordinary differential equation* because it contains the second time-derivative $\ddot{y}_{A,\mathrm{tot}}$. In contrast to ODEs that we encountered in Ch. 5 it cannot easily be solved by integration. Instead, we solve the ODE by first finding its particular solution and then its homogeneous solutions.

Conveniently, an undriven mechanical resonator always has a static solution with $\ddot{y}_{A,\mathrm{tot}} = 0$ and $y_{A,\mathrm{tot}} =$ constant as particular solution. Therefore, to solve

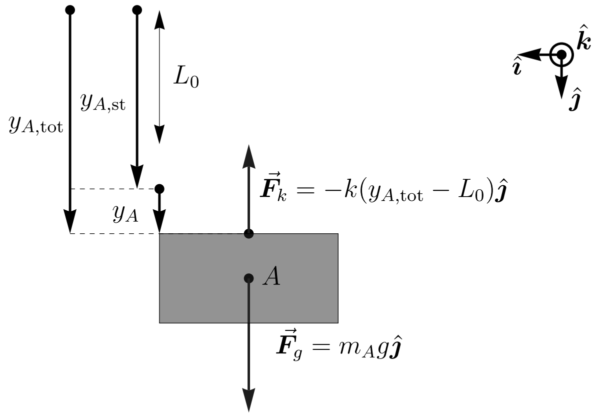


Figure 13.2: Free body diagram of the mass in a mass-spring system from Fig. 13.1.

free vibration problems, we always first find this static particular solution and then determine the dynamic part of the motion by solving the homogeneous ODE. This proceeds along the following steps:

1. Set the acceleration, velocity and other terms to zero and solve the EoM for the static equilibrium position $y_{A,\mathrm{tot}} = y_{A,\mathrm{st}}$. This particular solution is identical to the statics solution. If there are time-dependent functions set them to their time-averaged value (so a term $\cos \omega t$ becomes zero).
2. Introduce a new coordinate $y_A(t)$ which has its datum (zero point) at the static equilibrium point.
3. Substitute $y_{A,\mathrm{tot}} = y_{A,\mathrm{st}} + y_A(t)$ into the EoM. Then all static terms will cancel out, and you obtain a 'dynamic' homogeneous ODE which only contains $y_A(t)$ and can be solved.
4. If you already know the equilibrium position, you can skip steps 1-3 above and define the coordinate y_A with respect to that position, and directly write down the dynamic EoM without the constant terms, since you know that they will cancel out.

Let us illustrate this procedure.

13.2.1 Determining the static equilibrium position

We first solve for the particular static solution $y_{A,\text{tot}}(t) = y_{A,\text{st}}$ for which $\ddot{y}_{A,\text{tot}} = 0$ and $y_{A,\text{st}}$ is a constant. Substitution in Eq. (13.1) results in:

$$m_A g - k(y_{A,\text{st}} - L_0) = 0 \tag{13.2}$$

$$y_{A,\text{st}} = \frac{m_A g}{k} + L_0 \tag{13.3}$$

Thus we have found the static, time-independent solution $y_{A,\text{tot}}(t) = y_{A,\text{st}}$ of the ODE for which the y-coordinate of the mass is $y_{A,\text{st}}$ at all times. This position is called the *static equilibrium*, *equilibrium* or *rest* position of the mass. In this case, the static y-coordinate is the sum of the distance $\frac{m_A g}{k}$ the spring is elongated by the gravitational force and the relaxed length L_0 of the spring.

13.2.2 Equation of motion with respect to equilibrium

The static solution with $\ddot{y}_{A,\text{tot}} = 0$, is a particular solution of the ODE. However, it is not the only solution of the EoM, so let us now look for time-dependent solutions of the ODE. We define a time-dependent function $y_A(t)$ which represents the displacement of the mass *with respect to the static equilibrium position.* The function $y_A(t)$ is added to the found static solution such that the total displacement can be written as:

$$y_{A,\text{tot}} = y_A(t) + y_{A,\text{st}} \tag{13.4}$$

If we substitute this function $y_{A,\text{tot}}$ into equation Eq. (13.1), we find:

$$m_A g - k(y_A + y_{A,\text{st}} - L_0) = m_A \ddot{y}_A \tag{13.5}$$

$$-k y_A = m_A \ddot{y}_A \tag{13.6}$$

Here we used Eq. (13.2) to eliminate $y_{A,\text{st}}$. All constant, time-independent, terms in the equation always sum up to zero, like they do in static equilibrium. By rewriting Eq. (13.6) we find the equation of motion for free undamped vibrations in the form of the following ordinary differential equation (ODE):

Concept. *Equation of motion for free undamped vibrations*
The equation of motion is a homogeneous second order linear ODE, that holds at all times.

$$m_A \ddot{y}_A + k y_A = 0. \tag{13.7}$$

13.2.3 Solving the equation of motion

In Ch. 5 we discussed how first-order ordinary differential equations (ODE) can be solved by integration. The ODE in Eq. (13.7) can however cannot be easily solved by integration. Instead the strategy[1] to solve this ODE is to find a trial solution $x_A(t)$ of the right form with unknown parameters, substitute it into the ODE and use the ODE and initial conditions to determine the parameters. Since the ODE in Eq. (13.7) contains y_A and its second derivative $\ddot{y}_A$ it is logical to look for a trial solution that is a constant times its own second derivative, like $\cos(\omega_n t)$ or $e^{\lambda t}$.

Before starting with the solution, it is important to note that Eq. (13.7) is a homogeneous linear differential equation, because it only sums linear functions of y_A and its time-derivatives, which are equal to zero. These homogeneous linear ODE have the following useful property: if we find two different solutions of this ODE, $y_{A,1}(t)$ and $y_{A,2}(t)$, then substituting each of them individually in Eq. (13.7) gives zero, so their sum $y_A = y_{A,1} + y_{A,2}$ will also be a solution of the ODE, and summing many times is the same as multiplying, so $y_A = c_1 \times y_{A,1}$ is also a solution. These summative and multiplicative properties of the solutions of the ODE simplify the analysis of the motion significantly. Moreover, they allow solving the EoM with complex numbers, since if a complex function $y_A = y_{A,r} + iy_{A,i}$ is a solution of the ODE, the functions $y_{A,r}$ and $y_{A,i}$, which are the real and imaginary part of y_A, are also solutions of the ODE individually.

These ODEs can either be solved by trigonometric functions, or by complex functions. If you are familiar with working with complex numbers, they often make it simpler to solve the ODE. However, to make the reader familiar with both methodologies, we provide both the trigonometric and complex solutions in the derivation.

Derivation. *Free undamped vibration of a mass spring system*

Since we look for a function that is proportional to its own second derivative, we use a trial solution of the form $y_A(t) = A\cos(\omega_n t + \varphi_0) = \Re A_c e^{i\omega_n t}$ where $\Re$ stands for the real part of the function and $A_c = Ae^{i\varphi_0}$. The value of the parameters A, φ_0 and ω_n are unknown at this stage. Substituting this trial solution into the ODE Eq. (13.1) we obtain:

[1] Although solving this kind of second-order ODE is part of most introductory calculus courses, we discuss the procedure here for clarity.

$$
\begin{aligned}
m_A\ddot{y}_A + ky_A &= 0 && (13.8)\\
\underset{\textit{Trigonometric method}}{(-m_A\omega_n^2 + k)A\cos(\omega_n t + \varphi_0)} &= 0 && (13.9)\\
\underset{\textit{Complex method}}{(-m_A\omega_n^2 + k)\Re A_c e^{i\omega_n t}} &= 0 && (13.10)\\
(k - \omega_n^2 m_A)y_A &= 0 && (13.11)
\end{aligned}
$$

Here we used that $\dot{y}_A = -A\omega_n \sin(\omega_n t + \varphi_0)$ and $\ddot{y}_A = -A\omega_n^2 \cos(\omega_n t + \varphi_0)$, and similarly $\dot{y}_A = \Re[iA_c\omega_n e^{i\omega_n t}]$ and $\ddot{y}_A = \Re[-A_c\omega_n^2 e^{i\omega_n t}]$. From Eq. (13.11) we see that the proposed trial function is a solution if $y_A = 0$, and if $(k - m_A\omega_n^2) = 0$. The solution $y_A = 0$ is the static particular equilibrium solution that we derived before, so we now find the solution of the EoM.

Concept. *General solution $y_A(t)$ of the equation of motion for free undamped vibration:*

$$
\begin{aligned}
y_A(t) &= A\cos(\omega_n t + \varphi_0) = \Re A_c e^{i\omega_n t} && (13.12)\\
k - \omega_n^2 m_A &= 0 && (13.13)\\
\omega_n &= \sqrt{\frac{k}{m_A}} \quad \text{Natural angular resonance freq.} && (13.14)
\end{aligned}
$$

The motion for free undamped vibration $y_A(t)$ is plotted in Fig. 13.3.

13.2.4 Resonance frequency, period and phase

So we have found that the trial solution only satisfies the ODE for a specific value of $\omega_n = \sqrt{k/m_A}$, which is called the *natural angular resonance frequency* of the system and tells us something about the period of the vibration. The *period* T after which the vibration repeats itself is found from $\cos(\omega_n t + 2\pi) = \cos(\omega_n(t + T))$ from we find

$$
\begin{aligned}
T &= \frac{2\pi}{\omega_n} \quad \text{Period} && (13.15)\\
f_n &= \frac{1}{T} = \frac{\omega_n}{2\pi} \quad \text{Natural resonance frequency} && (13.16)
\end{aligned}
$$

Note that the found expression y_A is a solution of the ODE for any value of the *complex amplitude* A_c, which can be written as $A_c = Ae^{i\varphi_0}$, where the real

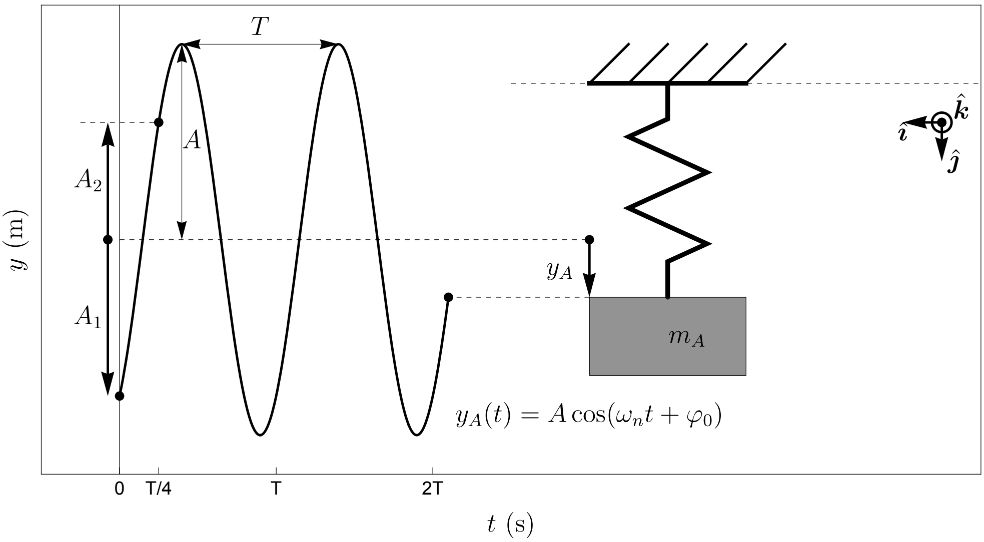


Figure 13.3: The free undamped vibration $y_A(t)$ of a mass-spring system and its period T, amplitude A and the constants A_1 and A_2 indicated.

number $A = |A_c|$ is the *amplitude* or *magnitude* of the vibration and the real number φ_0 is its *phase* at $t = 0$. In some cases it is more convenient to write the solution as a sum of a cosine and sine function with prefactors A_1 and A_2 as follows:

$$y_A(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \tag{13.17}$$

It can be shown that this equation is identical to that found in Eq. (13.12) by substituting $A_c = e^{i\varphi_0}$ and using Euler's equation $e^{ix} = \cos x + i \sin x$ and taking the real part of the complex function:

$$\begin{aligned}
y_A(t) &= \Re A[\cos(\varphi_0) + i\sin(\varphi_0)][\cos(\omega_n t) + i\sin(\omega_n t)] && (13.18)\\
A_1 &= A\cos(\varphi_0) && (13.19)\\
A_2 &= -A\sin(\varphi_0) && (13.20)\\
A &= \sqrt{A_1^2 + A_2^2} \quad \text{Amplitude} && (13.21)\\
\varphi_0 &= -\arctan\left(A_2/A_1\right) \quad \text{Phase at } t = 0 && (13.22)
\end{aligned}$$

The last four equations, were obtained by comparing Eq. (13.17) and Eq. (13.18). They can be used to relate the amplitude A and phase φ_0 to A_1 and A_2 and vice versa. In Fig. 13.4 the relations between the motion $y_A(t)$ and these four constants are visualised by a point moving along a circle with radius A.

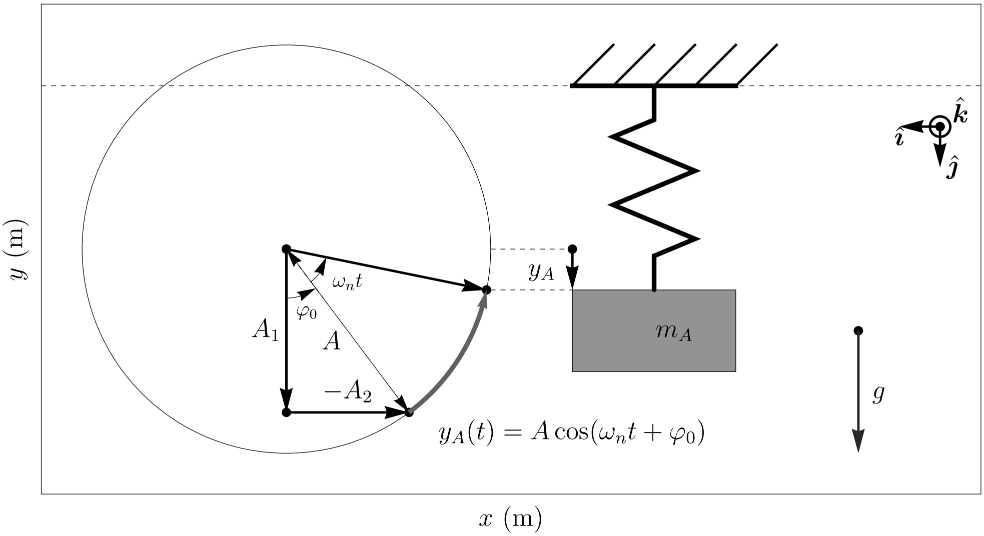


Figure 13.4: The free undamped vibration $y_A(t)$ of a mass-spring system visualised by a point moving at angular velocity ω_n along a circle with radius A (green arrow). The initial angle ϕ_0 at $t = 0$ and its relation to A_1 and A_2 is indicated.

13.2.5 Initial conditions

In the previous section we have found the solution of the ODE, however there are still two unknown constants A and φ_0 (or A_1 and A_2) that need to be found to know $y_A(t)$ at all times. Similar to the time integration in Eq. (5.34), we now also need initial conditions $y_A(t_1) = y_1$ and $\dot{y}_A(t_1) = v_1$ to determine the unknown constants as follows:

1. Determine the function $y_A(t)$ and its time derivative $\dot{y}_A$.
2. Substitute t_1 and write down the two equations $y_A(t_1) = y_1$ and $\dot{y}_A(t_1) = v_1$.
3. Solve the two equations for the two unknowns, A_1 and A_2 or A and φ_0.

S **Example 13.1** As an example let us consider the case where the initial conditions, position $y_A(0) = y_0$ and velocity $\dot{y}_A(0) = v_0$ at $t_1 = 0$ are known. We choose to use Eq. (13.17), $y_A(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$ and obtain the constants A_1 and A_2 as follows:

$$\begin{aligned} y_A(0) &= A_1 = y_0 && (13.23)\\ \dot{y}_A(t) &= -\omega_n A_1 \sin \omega_n t + \omega_n A_2 \cos \omega_n t && (13.24)\\ \dot{y}_A(0) &= \omega_n A_2 = v_0 && (13.25)\\ A_2 &= v_0/\omega_n && (13.26) \end{aligned}$$

By combining this with Eq. (13.4) and Eq. (13.17) we find for the case that the initial position y_0 and velocity v_0 are known that:

$$\begin{aligned} y_{A,\mathrm{tot}}(t) &= A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) + y_{A,\mathrm{st}} && (13.27)\\ y_{A,\mathrm{tot}}(t) &= y_0 \cos(\omega_n t) + v_0/\omega_n \sin(\omega_n t) + y_{A,\mathrm{st}} && (13.28) \end{aligned}$$

If required A and ϕ_0 can be determined using Eq. (13.21) and Eq. (13.22)

13.3 Free damped vibrations

In practice a vibration is never completely undamped, since there are always small practical imperfections that cause some kind of dissipation due to non-conservative forces. The most common type of damping, that is also most easily analysed, is by a velocity proportional force $\vec{\boldsymbol{F}}_c = -c\vec{\boldsymbol{v}}$, that is generated by a piston or other linear damper element. The force points in the opposite direction from the velocity vector of the mass, and thus the power done by such a force on the mass is given by $P = \vec{\boldsymbol{F}}_c \cdot \vec{\boldsymbol{v}} = -c|\vec{\boldsymbol{v}}|^2$, which is always negative. This causes the kinetic and potential energy of the mass-spring system to reduce to zero.

When connecting a linear damper to the mass, in parallel to the spring, as shown in Fig. 13.5, we obtain a mass-spring-damper system, we have from Newton's second law that $\vec{\boldsymbol{F}}_k + \vec{\boldsymbol{F}}_c = m_A\vec{\boldsymbol{a}}_A$, which when projected along the y-axis, results in the following equation of motion:

Concept. *Equation of motion for free damped vibrations with respect to the equilibrium position*

$$\begin{aligned} \sum F_y = -ky_A - c\dot{y}_A &= m_A\ddot{y}_A && (13.29)\\ m_A\ddot{y}_A + c\dot{y}_A + ky_A &= 0 && (13.30) \end{aligned}$$

Here we note that normally m_A, c and k are all positive real constants and in static equilibrium $\dot{y}_A = 0$, such that the damper does generate force and does not affect $y_{A,\mathrm{st}}$.

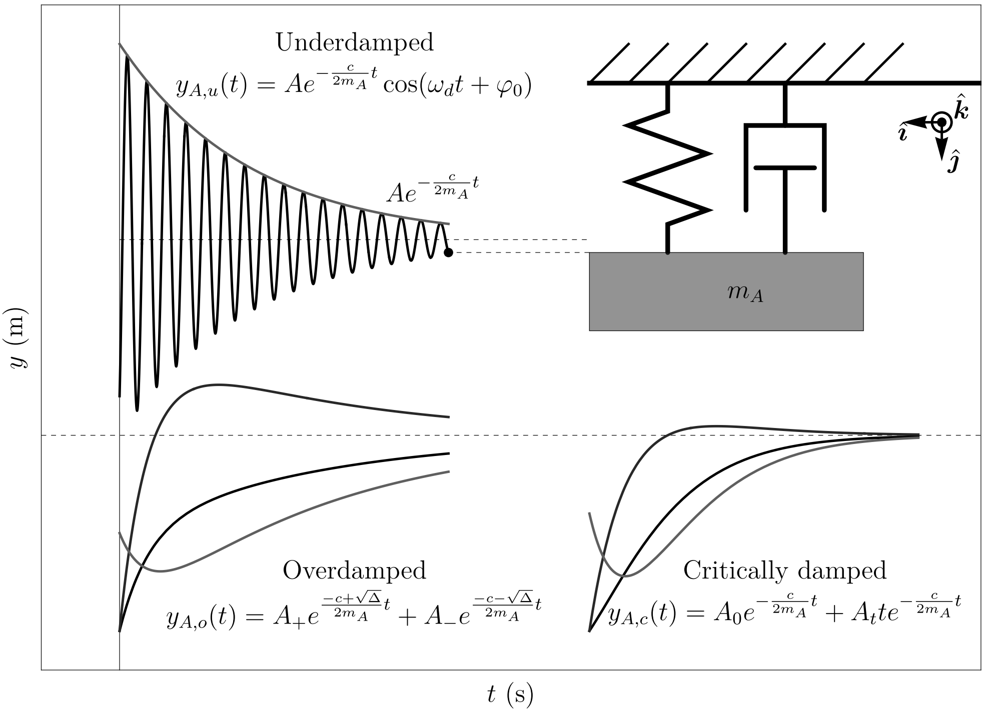


Figure 13.5: Examples of underdamped, overdamped and critically damped free vibrations. For overdamped and critically damped vibrations, different initial conditions are shown for the same ratio c/m_A.

Derivation. *Solving the EoM for free damped vibrations*

To solve this equation of motion we propose the following complex trial function:

$$y_A(t) = \Re A_c e^{\lambda t}. \tag{13.31}$$

In this equation both A_c and λ are complex numbers with a nonzero real and imaginary part. Note that we only use a complex trial function and don't use a trial function with trigonometric equations since working with complex numbers is a lot easier when working with damped vibrations. Substitution in Eq. (13.30) results in:

$$\Re\left(\left[m_A\lambda^2 + c\lambda + k\right] A_c e^{\lambda t}\right) = 0. \tag{13.32}$$

Since this equation has to hold for all times t, the quadratic equation in the square brackets has to be zero, and can be solved, to determine λ, using the quadratic formula or 'ABC formula', with $A = m_A$, $B = c$ and $C = k$:

$$m_A\lambda^2 + c\lambda + k = 0 \tag{13.33}$$

$$\lambda = \frac{1}{2A}\left(-B \pm \sqrt{B^2 - 4AC}\right) \tag{13.34}$$

$$\lambda = \frac{1}{2m_A}\left(-c \pm \sqrt{c^2 - 4m_A k}\right) = \frac{-c \pm \sqrt{\Delta}}{2m_A} \tag{13.35}$$

The term $\Delta = c^2 - 4m_A k$ is called the *discriminant* and depending on the sign of Δ, we obtain three different values of λ that correspond to three quite different types of solutions:

1. $\Delta > 0$: *Overdamped vibration*, the values of λ are both real and negative numbers with two solutions $\lambda_{o,\pm} = \frac{-c\pm\sqrt{\Delta}}{2m_A}$.
2. $\Delta < 0$: *Underdamped vibration*, the values of λ are complex numbers with a negative real part $-c/(2m_A)$. When taking the square-root of a negative number we get $\sqrt{-1} = i$, such that:

$$\frac{\sqrt{\Delta}}{2m_A} = i\frac{\sqrt{4m_A k - c^2}}{2m_A} = i\sqrt{\omega_n^2 - \frac{c^2}{4m_A^2}} \equiv i\omega_d \tag{13.36}$$

On the right side of this equation we have defined the *damped angular resonance frequency* $\omega_d \equiv \sqrt{\omega_n^2 - c^2/(4m_A^2)}$, and it can be seen that for small damping c, this damped resonance frequency is almost equal to the natural angular resonance frequency ($\omega_d \approx \omega_n$). We then obtain two values for λ that solve the EoM: $\lambda_{u,\pm} = \frac{-c}{2m_A} \pm i\omega_d$.

3. $\Delta = 0$: *Critically damped vibration.* In this case there is only one value $\lambda = -c/(2m_A)$. The value of the damping coefficient that belongs to this situation $\Delta = c_c^2 - 4m_A k = 0$ is called the *critical damping coefficient* $c_c = 2\sqrt{m_A k}$. For $c > c_c$ we have overdamped vibrations and for $c < c_c$ we get underdamped vibrations.

Let us now determine the different solutions Eq. (13.31) of the EoM with these values of λ.

13.3.1 Overdamped motion

Since λ is a real number for $\Delta > 0$, we need to add the solutions of the EoM for both values $\lambda_{o,\pm} = \frac{-c\pm\sqrt{\Delta}}{2m_A}$ to get a function that satisfies the equation of motion, while introducing two real constants A_+ and A_- that can be adjusted to satisfy the initial position and velocity conditions. We find:

$$y_{A,o}(t) = A_+ e^{\lambda_{o,+}t} + A_- e^{\lambda_{o,-}t} \tag{13.37}$$

$$y_{A,o}(t) = A_+ e^{\frac{-c+\sqrt{\Delta}}{2m_A}t} + A_- e^{\frac{-c-\sqrt{\Delta}}{2m_A}t} \tag{13.38}$$

13.3.2 Underdamped motion

For the case $\Delta < 0$, the complex number $A_c = Ae^{i\varphi_0}$ can be substituted in Eq. (13.31). Using the positive value[2] $\lambda_{u,+} = \frac{-c}{2m_A} + i\omega_d$ we obtain the solution:

$$y_{A,u}(t) = Ae^{-\frac{c}{2m_A}t}\Re e^{i(\omega_d t+\varphi_0)} = Ae^{-\frac{c}{2m_A}t}\cos(\omega_d t + \varphi_0) \tag{13.39}$$

The motion consists of a fast oscillating cosine function with a slowly decaying amplitude given by the exponential function. Just like in Sec. 13.2.4 one might also choose to replace $\cos(\omega_d t + \varphi_0)$ by the sum of a cosine and sine function, in which case the solution is written as:

$$y_{A,u}(t) = e^{-\frac{c}{2m_A}t}\left[A_1\cos(\omega_d t) + A_2\sin(\omega_d t)\right] \tag{13.40}$$

13.3.3 Critically damped motion*

For the general solution of critically damped motion we have a problem, since we have found only one value of λ that corresponds to the real solution

[2]Note that we could just as well have chosen $\lambda_{u,-}$, but that would have had the same effect as inverting the sign of φ_0 because $\cos x = \cos(-x)$. For consistency with earlier definitions we choose the positive sign.

$y_A = A_0 e^{-c/(2m_A)t}$. For this specific case it can by shown, as can be checked by substitution in Eq. (13.30), that the function $y_A = A_t t e^{-c/(2m_A)t}$ also obeys the EoM, such that the general solution for critically damped vibrations becomes:

$$y_{A,c}(t) = A_0 e^{-\frac{c}{2m_A}t} + A_t t e^{-\frac{c}{2m_A}t} \tag{13.41}$$

As can be seen in Fig. 13.5 the critically damped vibrations approach zero faster than the overdamped vibrations with the same damping c/m_A.

13.3.4 More complex free vibrations

Up to now we only dealt with mass-spring and mass-spring-damper systems. However, more complex systems can also result in the same equation of motion. For instance, applying Euler's second law to rotating rigid bodies connected to springs leads to an EoM that is similar to that of the free vibration of a mass-spring-damper system. Even though the constants and variables in that EoM are different from y_A, m_A, k and c, the EoM can be written similarly by introducing *effective* functions y_e, m_e, k_e and c_e, such that the EoM becomes:

$$m_e \ddot{y}_e + c_e \dot{y}_e + k_e y_e = 0. \tag{13.42}$$

Here, the function $y_e(t)$ can represent a position coordinate but can also be an angle, or any generalised coordinate that uniquely describes the position or orientation of the system. Mathematically, the motion of these more complex systems can be derived in exactly the same way as a mass-spring-damper system, like discussed in previous sections. Only the constants m_e, c_e and k_e and function name $y_e(t)$ are different. So, everything we have derived for the mass-spring-damper system remains valid for these more complex systems.

13.3.5 Free vibrations: solution procedure

Finally, we discuss step-by-step the procedure for analysing free vibrations discussed in this section and the previous one.

1. Sketch the system and coordinate system.
2. Draw the free-body diagram and project the force and/or moment vectors on the coordinate axes. Determine the relevant constraint equations.
3. Use Newton's second law and Euler's first law for translations and Euler's second law for rotations to determine the scalar equation of motion along the relevant translation or rotation coordinate.
4. Determine the static equilibrium position $y_{A,\text{st}}$ by setting all time derivatives in the EoM to zero $\ddot{y}_{A,\text{tot}} = \dot{y}_{A,\text{tot}} = 0$ and solving for y_A.

5. Define the time dependent part of the motion as $y_A(t) = y_{A,\text{tot}} - y_{A,\text{st}}$.
6. The system will only show free vibration if at least one of the EoMs can be written in the form $m_e\ddot{y}_e + c_e\dot{y}_e + k_e y_e = 0$.
7. Determine the equations for the variable y_e and constants m_e, c_e and k_e. Now one can either solve the ODE, like in the next step 8, or directly go to step 9 and use the known solutions.
8. To solve the ODE, substitute the trial function $y_e = \Re A_c e^{\lambda t}$ and its time-derivatives into the EoM to obtain $-m_e\lambda^2 + c_e\lambda + k_e = 0$. Solve for λ and substitute it back into the trial function to find the motion $y_e(t)$ like explained in the previous sections.
9. If $c_e = 0$, then we are dealing with free *undamped* vibration. In that case the solution of the EoM is $y_e(t) = \Re A e^{i\omega_n t+\varphi_0} = A\cos(\omega_n t + \varphi_0)$, with natural resonance frequency $\omega_n = \sqrt{k_e/m_e}$.
10. If $c_e \neq 0$ we are dealing with free *damped* vibration.
 Determine the discriminant $\Delta = c_e^2 - 4m_e k_e$.
 - If $\Delta > 0$ we are dealing with overdamped free vibration with $y_e = A_+e^{\lambda_+ t} + A_-e^{\lambda_- t}$ and $\lambda_\pm = \frac{-c_e\pm\sqrt{\Delta}}{2m_e}$.
 - If $\Delta < 0$ we are dealing with underdamped free vibration with $y_e = Ae^{-\frac{c_e}{2m_e}t}\cos(\omega_d t + \varphi_0)$ and $\omega_d = \sqrt{\omega_n^2 - c_e^2/(4m_e^2)}$.
 - If $\Delta = 0$ we are dealing with critically damped free vibration with $y_e = A_0 e^{-\frac{c_e}{2m_e}t} + A_t t e^{-\frac{c_e}{2m_e}t}$.
11. Two initial conditions of the form $y_e(t_1) = y_1$ or $\dot{y}_e(t_2) = v_2$ need to be solved to determine the two unknown constants in the solution (A & φ_0, A_+ & A_- or A_0 & A_t). See Sec. 13.2.5.
12. In some cases it is convenient to rewrite the cosine with phase-shift φ_0 into a sum of a cosine and sine using the mathematical equation:

$$A\cos(\omega t + \varphi_0) = A_1\cos(\omega t) + A_2\sin(\omega t), \tag{13.43}$$

 where $A^2 = A_1^2 + A_2^2$ and $\tan(\varphi_0) = -A_2/A_1$.

S **Example 13.2** Consider the car A, in Fig. 13.6 with a mass of m=1000 kg that can be considered as a point mass. It has a spring stiffness of k=50000 N/m and a damping constant of $c = 1000$ N·s/m. The car drops from a small height, such that it reaches its equilibrium position $y_A(0) = 0$ with a speed of $v_0 = -1$ m/s at $t = 0$.

- *Determine the motion $y_A(t)$ of the car.*

We follow the steps from this section. The sketch and FBD are given in Fig. 13.6.

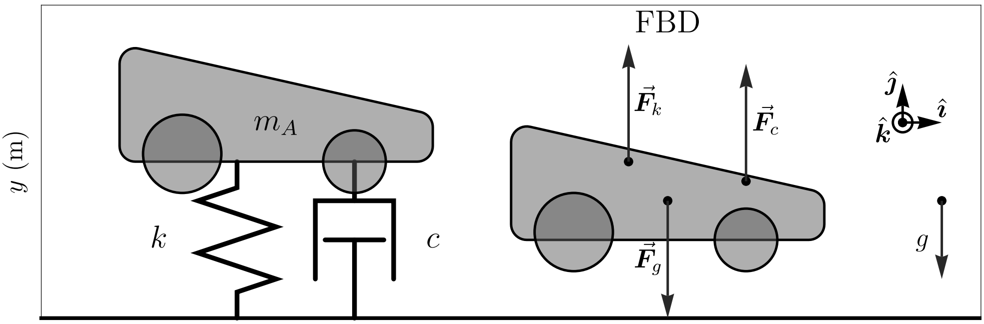


Figure 13.6: Free damped vibration of a car suspension.

So, we project the forces and write down the equation of motion along the y-axis:

$$\sum F_y = F_{k,y} + F_{c,y} + F_{g,y} = m_A \ddot{y}_{A,\text{tot}} \tag{13.44}$$

$$-k(y_{A,\text{tot}} - L_0) - c\dot{y}_{A,\text{tot}} - m_A g = m_A \ddot{y}_{A,\text{tot}} \tag{13.45}$$

$$-k y_A - c\dot{y}_A = m_A \ddot{y}_A \tag{13.46}$$

$$m_A \ddot{y}_A + k y_A + c\dot{y}_A = 0 \tag{13.47}$$

Here we used that $y_{A,\text{st}} = -\frac{m_A g}{k} + L_0$ like in Eq. (13.2), and $y_{A,\text{tot}} = y_A + y_{A,\text{st}}$ to eliminate the static particular solution. Although, since we are only asked for the motion with respect to the equilibrium position, we could also have skipped this step, and directly eliminate the static terms. Now we see that the equation of motion Eq. (13.47) can be written in the form of Eq. (13.42), with $y_e = y_A$, $m_e = m_A$, $c_e = c$ and $k_e = k$ (we will encounter more complex cases later). Now we can solve this EoM, like in Sec. 13.3. We will not repeat the derivation and directly substitute the values of m_e, c_e and k_e in the solution. We first determine that the discriminant $\Delta = c^2 - 4m_A k = -199 \times 10^6$ kg^2/s^2 is negative, such that we deal with an underdamped vibration. Then we determine the damped resonance frequency as $\omega_d = \sqrt{k_e/m_e - c_e/(4m_e^2)}$. The solution can then be written as:

$$y_A(t) = A e^{-(c_e/2m_e)t} \cos(\omega_d t + \varphi_0) \tag{13.48}$$

Finally we need to determine the constants A and φ_0 from the initial conditions. Since $y_A(t = 0) = 0$, we find that $\varphi_0 = \pi/2$. By taking the time derivative of y_A, with the product rule and using that $\dot{y}_A(0) = v_0$ we find:

$$\begin{aligned} \dot{y}_A(0) &= [-(c_e/2m_e)\cos(\omega_d 0 + \pi/2) - \omega_d \sin(\omega_d 0 + \pi/2)]\, A e^{-(c_e/2m_e)0} = v_0 \\ &= -\omega_d A = v_0 = -1 \text{ m/s}, \end{aligned} \tag{13.49}$$

from which we find A. Thus we have fully determined all constants in Eq. (13.48), and fully determined the motion $y_A(t)$.

A **Example 13.3** For the car in the previous example we now design the optimal damping, that reduces the time for the amplitude to go to zero. The initial speed v_0 and kinetic energy of the car is kept constant and the damping coefficient c is varied from underdamped $c = \frac{1}{2}c_c$, to critically damped $c = c_c$ to overdamped $c = 2c_c$. *Plot the motion of the car for these three situations, and determine the optimal value of c.*
First we determine the constants $\omega_d = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$, $c_c = 2\sqrt{km}$ and $\Delta = c^2 - 4mk$, and define $\omega_{do} = \sqrt{\Delta}/(2m)$. Then we use the initial conditions $y_A(0) = 0$ and $\dot{y}_A(0) = v_0 = -1$ m/s, to determine the constants A_* and φ_0 for the underdamped, critically damped and overdamped motion. Check with some math that $\phi_0 = \Pi/2$ this results in:

$$y_{A,u}(t) = v_0 t e^{-\frac{c}{2m}t} \frac{\sin(\omega_d t)}{\omega_d t} \tag{13.50}$$

$$y_{A,c}(t) = v_0 t e^{-\frac{c}{2m}t} \tag{13.51}$$

$$y_{A,o}(t) = v_0 t e^{-\frac{c}{2m}t} \frac{\sinh(\omega_{do} t)}{\omega_{do} t} \tag{13.52}$$

We see that for a damping coefficient $c = c_c$ the frequencies ω_d and ω_{do} approach zero and because for small x we have $\sin x \approx x$ and $\sinh x \approx x$, the three expressions become equal. In Fig. 13.7 it can be seen the that the solution approaches zero fastest when the damper is chosen to be critically damped with $c = c_c$. The underdamped motion damps slower because of the smaller value of c in the exponent, and the overdamped motion is slower due to the slow decay of the A_+ term in Eq. (13.38).

13.4 Forced vibrations

After having considered free vibrations we now turn to forced vibrations. In a forced vibration there is an additional time-dependent external force $\vec{\boldsymbol{F}}(t)$ acting on the mass. These forced vibrations are important in examples like clocks, vibrations in cars driving over a bumpy road, the beating of your heart, audio speakers, and electromechanical frequency generators for data transmission in your mobile phone.

Fig. 13.8 shows a sketch and FBD of a mass-spring-damper system driven by a time-dependent force $F(t)$ in the y-direction:

$$\vec{\boldsymbol{F}}(t) = F_0 \Re e^{i\omega t} \hat{\boldsymbol{\jmath}} = F_0 \cos(\omega t) \hat{\boldsymbol{\jmath}} \tag{13.53}$$

Similar to Eq. (13.1) we obtain the equation of motion by projection of the forces on the y-axis and using Newton's second law:

$$\sum F_y = m_A g - k(y_{A,\text{tot}} - L_0) - c\dot{y}_{A,\text{tot}} + F_0 \Re e^{i\omega t} = m_A \ddot{y}_{A,\text{tot}} \tag{13.54}$$

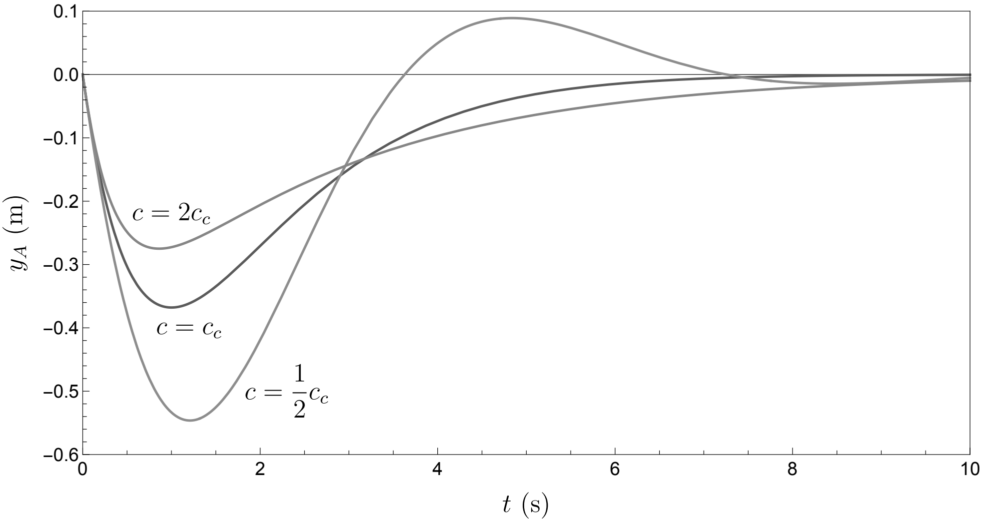


Figure 13.7: Free damped vibration of a car suspension with $\omega_n = 1$ rad/s, initial velocity $v_0 = -1$ m/s for various values of the damping coefficient c. The critically damped case with $c = c_c$ approaches the equilibrium position $y_A = 0$ fastest.

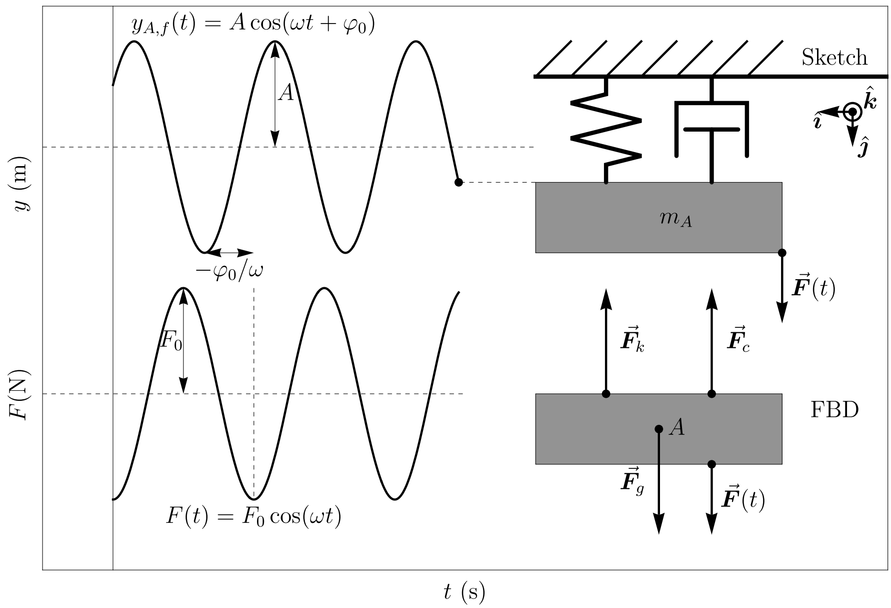


Figure 13.8: Sketch and FBD of a spring-mass-damper system driven (forced) by a time dependent force $\vec{F}(t)$. On the left the steady-state motion $y_A(t)$ of the point mass is shown, which is delayed by a time $\Delta t = -\varphi_0/\omega$ with respect to the force $F(t)$.

13.4.1 Determining the equilibrium position

Like for the free vibration we first determine the static equilibrium position by setting all time derivatives to zero: $\ddot{y}_{A,\mathrm{tot}} = \dot{y}_{A,\mathrm{tot}} = 0$. Note, that if the average value of the external force F_{avg} is not zero, it needs to be included in the determination of the static equilibrium position. However, for a cosine function $F_{\mathrm{avg}} = 0$. We find for the static equilibrium position in the presence of vibrations:

$$m_A g - k(y_{A,\mathrm{st}} - L_0) + F_{\mathrm{avg}} = 0 \quad (13.55)$$

$$y_{A,\mathrm{st}} = \frac{m_A g}{k} + L_0 + \frac{F_{\mathrm{avg}}}{k} \quad (13.56)$$

So for $F(t) = F_0 \cos \omega t$, the static equilibrium position $y_{A,\mathrm{st}}$ is identical to that found for free vibrations.

13.4.2 Equation of motion with respect to equilibrium

Similar to the case of free vibrations, we substitute $y_{A,\mathrm{tot}}(t) = y_A(t) + y_{A,\mathrm{st}}$ into Eq. (13.54), such that all static terms cancel and we obtain the EoM for the displacement $y_A(t)$ of the mass with respect to the equilibrium position:

Concept. *Equation of motion for forced damped vibrations*

$$\Re\left[m_A \ddot{y}_A + c\dot{y}_A + k y_A = F_0 e^{i\omega t}\right] \quad (13.57)$$

We see that the only difference between the EoM for forced and free vibrations is the forcing term $F_0 \Re e^{i\omega t}$.

13.4.3 Particular and homogeneous solutions of the EoM

Let us assume we find a particular solution $y_{A,f}(t)$ of the differential equation (13.57). Then we can add a solution $y_{A,h}(t)$ of the corresponding homogeneous EoM, $m_A\ddot{y}_A + c\dot{y}_A + k y_A = 0$. And then the resulting function $y_A(t) = y_{A,f}(t) + y_{A,h}(t)$ is still a solution of the EoM Eq. (13.57) as can be seen here:

$$\Re\left[m_A(\ddot{y}_{A,f} + \ddot{y}_{A,h}) + c(\dot{y}_{A,f} + \dot{y}_{A,h}) + k(y_{A,f} + y_{A,h}) = F_0 e^{i\omega t}\right] \quad (13.58)$$

The terms with $y_{A,f}(t)$ add up to $F_0 e^{i\omega t}$ and the terms with $y_{A,h}$ add up to zero. The function $y_{A,f}(t)$ is called the particular solution of the ODE and $y_{A,h}(t)$ is the homogeneous (or complementary) solution of the EoM for forced vibrations. We see that the differential equation for $y_{A,h}(t)$ is identical to that of a free damped vibration. So, we can just use the methods from the previous section to analyse these solutions. Moreover, due to the damping, these vibrations damp out after waiting for a sufficiently long time after which we are left with $y_{A,f}(t)$, which is therefore also called the *steady-state solution* of the forced damped vibration EoM, while $y_{A,h}(t)$ is called the *transient (part of the) solution*. We note that $y_{A,h}(t)$ has two free constants A_h and $\varphi_{0,h}$ that still need to be determined using the initial conditions. The procedure to determine these constants using the initial conditions is identical to that explained in section Sec. 13.2.5, although the equations for position and velocity are longer, since they contain both the steady-state and transient part of the solution.

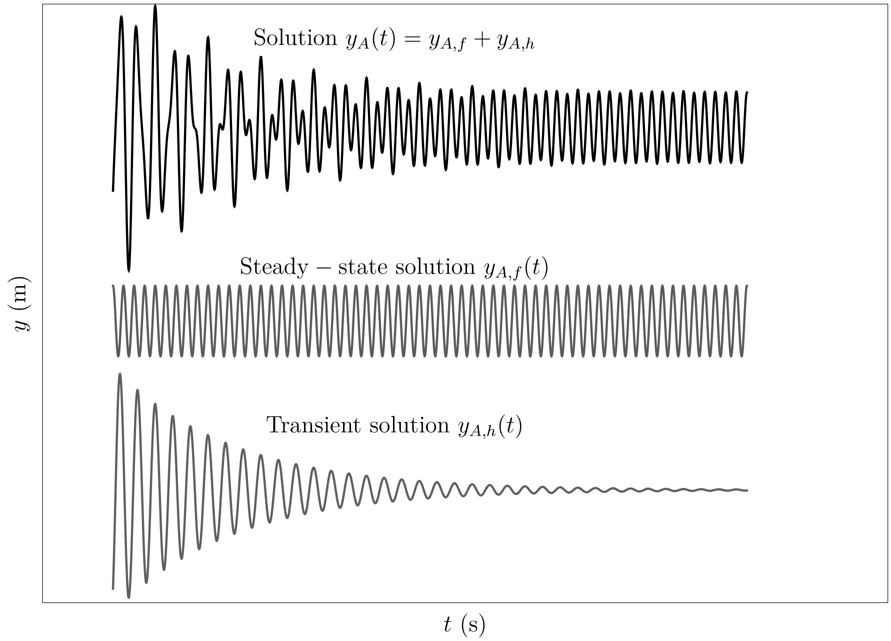


Figure 13.9: The general solution $y_A(t)$ of forced vibration is a sum of a steady-state solution $y_{A,f}$ and a transient solution $y_{A,h}$. Note that since the driving frequency ω can be different from the resonance frequency ω_d, beating effects (Fig. 2.1) are observed in the initial transient phase of the motion.

13.4.4 Solving the EoM for forced vibrations

Since the transient solutions $y_{A,h}(t)$ were already found from the free vibration analysis, we only need to solve the particular, steady-state solution $y_{A,f}(t)$.

Derivation. *Forced damped vibration*

To solve the EoM we are looking for an equation that is proportional to its own time-derivatives and also proportional to the forcing function. Therefore we attempt using a trial solution of the following form:

$$y_A(t) = \Re A_c e^{i\omega t} \tag{13.59}$$

Note that in contrast to the case of free vibrations, we do not solve for an exponent λ that gives us a (damped) natural frequency, but instead we take the forcing frequency ω directly into the trial function y_A. Let us now show that this trial function is indeed a solution of the differential equation and determine the complex amplitude A_c by substituting it into the EoM Eq. (13.57):

$$\Re\left[A_c e^{i\omega t}\left(-\omega^2 m_A + i\omega c + k\right) = F_0 e^{i\omega t}\right] \tag{13.60}$$

We see from this equation that our choice to choose $e^{i\omega t}$ as exponential function was correct, since we can now divide both sides of this equation by $e^{i\omega t}$ to eliminate it. Then the complex amplitude A_c needs to obey this equation to ensure $y_A(t)$ is a solution at all times:

$$A_c(k - \omega^2 m_A + i\omega c) = F_0 \tag{13.61}$$

By dividing both sides by the term between brackets we find the complex amplitude A_c of the vibration:

$$A_c = \frac{F_0}{k - \omega^2 m_A + i\omega c} \tag{13.62}$$

Now we would like to express the complex amplitude as a product of a real amplitude A and phase φ_0, such that $A_c = Ae^{i\varphi_0}$. We find for the amplitude:

$$A = |A_c| = \sqrt{A_c A_c^*} \tag{13.63}$$

$$= \left[\frac{F_0}{(k - \omega^2 m_A) + i\omega c} \cdot \frac{F_0}{(k - \omega^2 m_A) - i\omega c}\right]^{1/2} \tag{13.64}$$

$$= \frac{F_0}{\sqrt{(k - \omega^2 m_A)^2 + (\omega c)^2}} = \frac{F_0/m_A}{\sqrt{(\omega_n^2 - \omega^2)^2 + \frac{\omega^2\omega_n^2}{Q^2}}}, \tag{13.65}$$

where we used Eq. (2.34) and $Q = k/(\omega_n c)$. From this equation, that is plotted in Fig. 13.10 we see that for small values of damping c, the amplitude approximately has a maximum if the driving frequency ω is equal to the natural frequency $\omega_n = \sqrt{k/m_A}$, in this situation the system is driven at resonance.

The amplitude when driving at its natural frequency $\omega = \omega_n$ is $A = F_0Q/k$, while it is $A = F_0/k$ at low frequencies. The ratio between these two numbers is called the quality factor $Q = k/(\omega_n c)$ and is a measure of how many periods it takes for the resonant vibrations to dampen out if the driving force is removed. At critical damping $Q_c = \sqrt{km_A}/c_c = \frac{1}{2}$ and for the undamped system with $c = 0$ the amplitude and Q-factor becomes infinite when it is driven at resonance. At very high driving frequencies ω the amplitude tends to zero because the inertia of the mass prevents its motion to follow the driving force.

Near and above the resonance frequency ω_n, a phase difference φ_0 develops, such that the motion is delayed with respect to the driving force, as indicated in Fig. 13.8 and Fig. 13.11. The value of this phase shift can be found by taking the ratio of the imaginary and real parts of $A_c = Ae^{i\varphi_0}$ and Eq. (13.62):

$$\tan(\varphi_0) = \frac{\Im A_c}{\Re A_c} = \frac{-\omega c}{k - \omega^2 m_A} \tag{13.66}$$

We see that at low driving frequencies ω the motion is in phase with the force because the phase difference $\varphi_0 = 0$. At larger driving frequencies the phase difference φ_0 becomes negative, indicating that the motion lags behind the force. This continues up to the natural resonance frequency, where $\omega^2 = \omega_n^2 = k/m_A$ and $\varphi_0 = -\pi/2$, for which the motion is -90 degrees out of phase, and the velocity is in phase with the force. At even higher frequencies, the phase lag between motion and force increases more, until it reaches a phase difference close to -180 degrees with $\varphi_0 = -\pi$ at very high driving frequencies $\omega \gg \omega_n$.

Steady-state solution for damped forced vibrations

We have now fully determined the complex amplitude $A_c = Ae^{i\varphi_0}$, with A being given by Eq. (13.65) and φ_0 by Eq. (13.66), and thus we have found the function $y_{A,f}(t)$ in Eq. (13.59) and proven that it is a solution of the EoM in Eq. (13.57).

$$y_{A,f}(t) = \Re A e^{i(\omega t + \varphi_0)} = A\cos(\omega t + \varphi_0) \tag{13.67}$$

$$A = \frac{F_0/m_A}{\sqrt{(\omega_n^2 - \omega^2)^2 + \frac{\omega^2 c^2}{m_A^2}}} \tag{13.68}$$

$$\varphi_0 = \arctan\frac{-\omega c}{k - \omega^2 m_A} \tag{13.69}$$

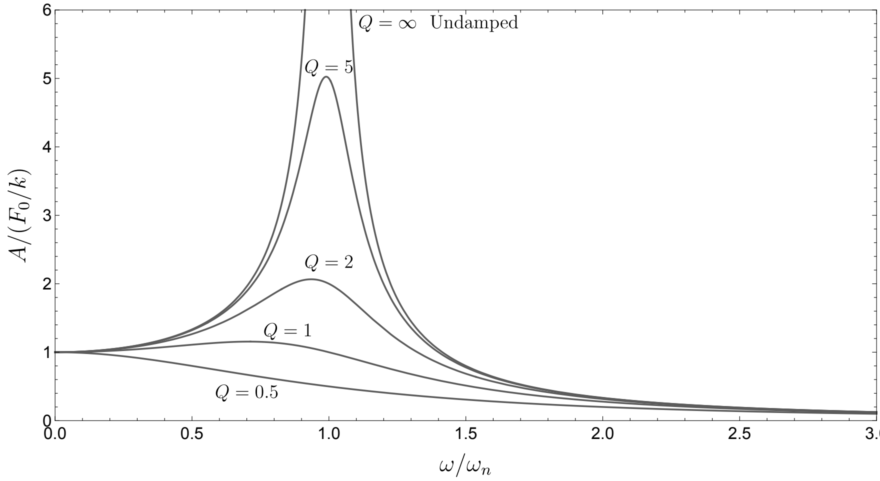


Figure 13.10: Amplitude A of the steady state motion $y_{A,f}$ as a function of the driving frequency ω. At the natural resonance frequency ω_n the amplitude rises approximately by a factor $Q = \frac{\sqrt{km}}{c}$.

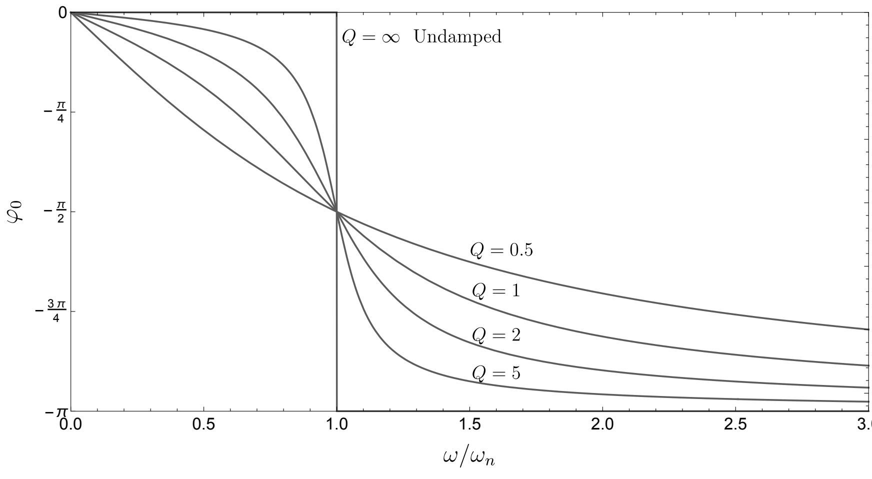


Figure 13.11: Phase φ_0 of the steady state motion $y_{A,f}$ with respect to the force. At low driving frequencies the motion is in-phase ($\varphi_0 = 0$) with the driving force. At resonance there is a phase lag of $\pi/2$ and at high frequencies, the phase lag is π.

In contrast to free damped vibrations, the steady-state solution Eq. (13.67) is the same for underdamped, overdamped and critically damped systems.

Steady state solution for forced undamped vibrations

An important special case is the situation in which there is no damper connected to the mass, such that $c = 0$. In this situation, the system will exhibit *undamped forced vibrations*. The solution of this EoM can be easily obtained from our previous analysis by just setting $c = 0$. It is seen from Eq. (13.65) and Eq. (13.66), that then $A = \frac{F_0}{m_A}|\omega_n^2 - \omega^2|^{-1}$ and $\varphi_0 = 0$ for $\omega < \omega_n$ and $\varphi_0 = -\pi$ for $\omega > \omega_n$. We therefore have for the solution of the EoM for forced undamped vibrations:

$$y_{A,fu}(t) = \frac{F_0}{m_A(\omega_n^2 - \omega^2)} \cos(\omega t) \tag{13.70}$$

A special feature of undamped vibration is that its amplitude goes to infinity if the driving frequency equals the natural resonance frequency ($\omega = \omega_n$), see Fig. 13.10 for $Q = \infty$. In practice such infinite amplitudes never occur and a small damping constant c is always present.

13.4.5 More complex systems involving forced vibrations

The same methodology we have just derived for mass-spring-damper systems, can be extended to rigid bodies and other more complex systems for which the kinetic analysis yields an EoM that has the form of Eq. (13.57):

$$\Re\left[m_e\ddot{y}_e + c_e\dot{y}_e + k_e y_e = F_{0,e}e^{i\omega t}\right] \tag{13.71}$$

For analyzing such systems one first determines the equations for y_e, $F_{0,e}$, m_e, c_e and k_e by comparing Eq. (13.71) to the EoM and then analyses the vibrations in an exactly identical way as for the mass-spring-damper system (see also Sec. 13.3.4).

13.4.6 Forced vibration: solution procedure

Based on the previous analysis, let us summarise the step-by-step procedure for analysing forced vibrations and obtaining their motion.

1. Sketch the system and coordinate system.
2. Draw the free-body diagram and project the force and/or moment vectors on the coordinate axes. Determine the relevant constraint equations.

3. Use Newton's second law and Euler's first law for translations and Euler's second law for rotations to determine and simplify the scalar equations of motion, reducing it to a single scalar EoM.
4. Determine the static equilibrium position $y_{A,\mathrm{st}}$ by setting all time derivatives in the EoM to zero $\ddot{y}_{A,\mathrm{tot}} = \dot{y}_{A,\mathrm{tot}} = 0$, setting the forcing term $F(t)$ to its time-averaged value F_{avg} and solving for y_A.
5. Define the time dependent part of the motion as $y_A(t) = y_{A,\mathrm{tot}} - y_{A,\mathrm{st}}$.
6. The system will show forced vibration if the EoM can be written in the form $m_e\ddot{y}_e + c_e\dot{y}_e + k_e y_e = F_{0,e}\cos(\omega t)$.
7. Determine the equations for y_e, $F_{0,e}$, m_e, c_e and k_e in terms of the given variables, constants or numerical values.
8. Substitute a trial function of the form $y_e = \Re A_c e^{i\omega t}$ into the EoM by taking the first and second derivative.
9. Obtain the equation $A_c(-m_e\omega^2 + ic_e\omega + k_e) = F_{0,e}$ and determine the complex steady-state amplitude A_c.
10. Use $A_c = Ae^{i\varphi_0}$ to convert the complex prefactor to a real amplitude A and phase φ_0.
11. If $c_e = 0$, then we are dealing with forced *undamped* vibration. In that case $\varphi_0 = 0$ such that the steady-state solution of the EoM is $y_e(t) = A\cos(\omega t)$, with $A = F_{0,e}/(k_e - m_e\omega^2)$.
12. If $c_e \neq 0$ we are dealing with forced *damped* vibration. The complex amplitude can be converted into a real amplitude A and phase φ_0 using the relation $A_c = Ae^{i\varphi_0}$.
    The steady-state solution of the EoM is: $y_{e,f}(t) = A\cos(\omega t + \varphi_0)$.
    The amplitude of the steady-state solution is:
    $A = F_{0,e}/\sqrt{m_e^2(\omega_n^2 - \omega^2)^2 + \omega^2 c_e^2}$.
    The phase difference between force and motion is:
    $\varphi_0 = \arctan[-\omega c_e/(k_e - \omega^2 m_e)]$.
13. Solutions $y_{e,h}(t)$ of the homogeneous EoM
    $m_e\ddot{y}_{e,h} + c_e\dot{y}_{A,h} + k_e y_{A,h} = 0$ for free vibrations can be summed to the steady state solution $y_{A,f}(t)$ to obtain new solutions of the EoM. For damped vibrations ($c > 0$), the function $y_{A,h}(t)$ reduces to zero after sufficient time, it is therefore called the transient part of the solution. In general, any solution of the forced EoM can always be expressed as $y_A(t) = y_{A,f}(t) + y_{A,h}(t)$.
14. Two initial conditions of the form $y_A(t_1) = y_1$ or $\dot{y}_A(t_2) = v_2$ need to be solved to determine the two unknown constants in the transient part of the solution $y_{A,h}(t)$ (see Sec. 13.2.5). Note that for forced vibrations the full motion $y_A(t)$ including both the steady-state and transient parts needs to be used to determine these constants.

13.4.7 More complex driving forces*

Up to now we have only dealt with a single cosine shaped forcing function. However, in practice more complex periodic forcing functions can act, which are a sum of multiple cosine functions, each with a different driving frequency ω_j and a different complex amplitude F_j, and with the total forcing function $F(t) = \Re \sum_{j=1}^{j=N} F_j e^{i\omega_j t}$, where there are N terms such that j runs from $j = 1 \dots N$. The methodology derived in this section can be used to find a solution $y_{A,j}$ for each of these driving frequencies and then add all solutions to find a solution for the full EoM. To satisfy the EoM each of these functions $y_{A,j}$ needs to satisfy an ODE of the form:

$$m_A \ddot{y}_{A,j} + c\dot{y}_{A,j} + ky_{A,j} = F_j e^{i\omega_j t} \tag{13.72}$$

From each of these N ODEs, with $y_{A,j} = A_{c,j} e^{i\omega_j t}$ we determine the complex motion amplitude $A_{c,j}$ using the corresponding complex force amplitude F_j and frequency ω_j with Eq. (13.62), $A_{c,j} = F_j/(k - \omega_j^2 m_A + i\omega_j c)$. By summing we find a steady-state solution of the form:

$$y_{A,f}(t) = \Re \left[\sum_j A_{c,j} e^{i\omega_j t} \right] \tag{13.73}$$

Since, according to Fourier analysis, any periodic time-dependent function can be written as $F(t) = \Re \sum_j F_j e^{i\omega_j t}$, this procedure allows to analyse the vibrations and motion of a system in response to any periodic force. Note that driving with two or more frequencies can result in beating, like shown in Fig. 2.1.

13.4.8 Vibrations of rotating rigid bodies

As mentioned in Sec. 13.4.5, everything discussed in this chapter can also be used to analyse vibrations of rotating rigid bodies. We illustrate this by an example which includes a massive pulley as rigid body and also shows how to reduce the equation of motion of multiple objects to a single one by constraint equations.

A **Example 13.4** *Description:*
The sketch of the problem is given in Fig. 13.12. A point mass B with mass m_B is suspended by a rope via a pulley A that has moment of inertia I_G, radius R and mass m_A (so it is different from the massless pulleys we have considered earlier). The other side of the rope is connected to a spring with stiffness k. The rope does not slip and is always tight under tension. When the system is in static equilibrium,

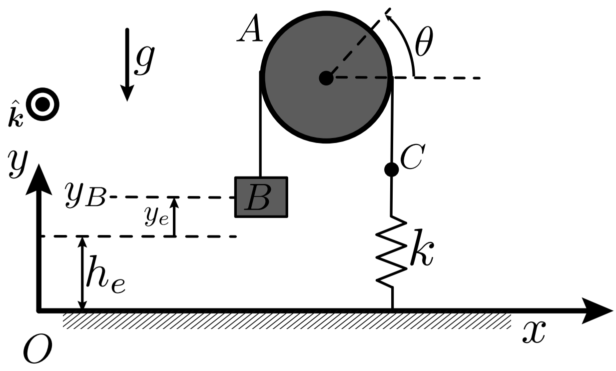


Figure 13.12: Sketch belonging to Example 13.4.

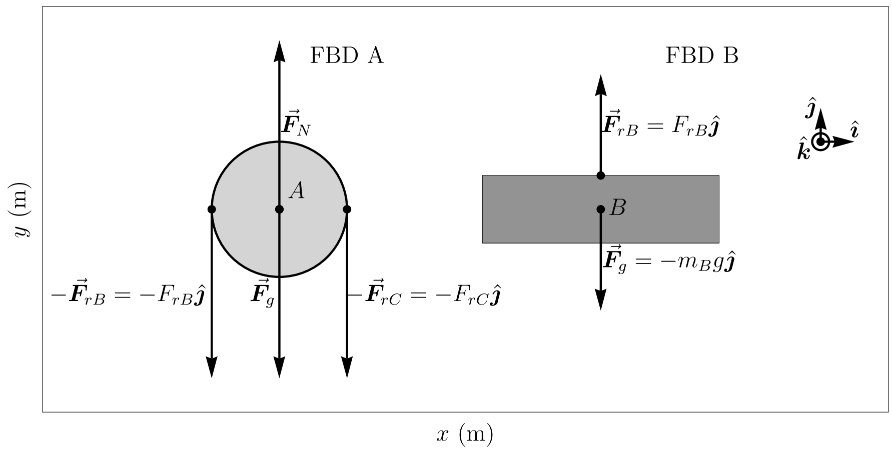


Figure 13.13: FBD belonging to Example 13.4.

the mass B is at height $y_B = h_e$. We define $y_e = y_B - h_e$ as the displacement of B with respect to this equilibrium position. θ_e is the angular displacement of the pulley with respect to the equilibrium position. y_C is the displacement of point C with respect to the equilibrium position.

Question: Determine the rotational motion $\theta_e(t)$ of the pulley A with respect to the equilibrium position when the system is released from rest at $\theta_e = \theta_{e0}$.

Solution:

- Draw the FBD of A and B shown in Fig. 13.13. Don't forget the normal force $\vec{\boldsymbol{F}}_N$ and gravitational force $\vec{\boldsymbol{F}}_g$ on the pulley. The rope generates forces $\vec{\boldsymbol{F}}_{rB}$ and $\vec{\boldsymbol{F}}_{rC}$ on points B and C respectively and by Newton's third law the opposite forces act on the pulley. Ask yourself: *why are the forces $\vec{\boldsymbol{F}}_{rB}$ and $\vec{\boldsymbol{F}}_{rC}$ different?* The answer is: they are different because a moment is needed to provide the angular acceleration of the disk of the pulley according to Euler's second law, since the pulley is not massless. If the forces would be equal this moment would be zero. This angular acceleration is provided via the static friction force between the disk and the rope, which does not slip. In the FBD we directly provide the equations to expand the vectors into their scalar components along the y-axis.
- Determine the constraint equations from the rope. Since the motion with respect to the equilibrium position is asked, we do not have to determine the static equilibrium position anymore. Since the rope has constant length and does not slip, and because in equilibrium all coordinates are zero $y_e = y_C = \theta_e = 0$, we have 2 constraint equations. We also write down the equation for the spring force F_{rC} of the rope on point C:

$$y_e = -\theta_e R \tag{13.74}$$

$$y_C = +\theta_e R \tag{13.75}$$

$$F_{rC} = k y_C \tag{13.76}$$

- Now we write down the equation of motion for mass B along the y-axis, with respect to equilibrium:

$$\sum F_{By} = F_{rB} = m_B \ddot{y}_e \tag{13.77}$$

  Note that the gravitational force is excluded, since it does not contribute to the time-dependent part of the force.
- It is seen that the motion of the pulley is a pure rotation around its centre. Due to the pin around which it rotates, there is no translation of its CoM, such that we only need to consider the rotational EoM of the pulley, i.e. Euler's

second law. As reference point we choose the centre of the pulley G.

$$\sum M_{Az} = I_G\ddot{\theta}_e\hat{\boldsymbol{k}} \tag{13.78}$$

$$\vec{\boldsymbol{r}}_{B/G} \times (-\vec{\boldsymbol{F}}_{rB}) + \vec{\boldsymbol{r}}_{C/G} \times (-\vec{\boldsymbol{F}}_{rC}) = I_G\ddot{\theta}_e\hat{\boldsymbol{k}} \tag{13.79}$$

$$(-R\hat{\boldsymbol{\imath}}) \times (-F_{rB}\hat{\boldsymbol{\jmath}}) + (R\hat{\boldsymbol{\imath}}) \times (-F_{rC}\hat{\boldsymbol{\jmath}}) = I_G\ddot{\theta}_e\hat{\boldsymbol{k}} \tag{13.80}$$

Now we use Eqs. (13.74)–(13.77) to simplify the EoM to a single scalar equation in θ_e:

$$(Rm_B\ddot{y}_e - Rky_C)\hat{\boldsymbol{k}} = I_G\ddot{\theta}_e\hat{\boldsymbol{k}} \tag{13.81}$$

$$-R^2m_B\ddot{\theta}_e - R^2k\theta_e = I_G\ddot{\theta}_e \tag{13.82}$$

$$(\frac{1}{2}m_A + m_B)R^2\ddot{\theta}_e + R^2k\theta_e = 0 \tag{13.83}$$

In the last step we used that the moment of inertia of a disk around its centre is $I_G = \frac{1}{2}mR^2$.

- We see that the EoM resembles a free undamped vibration $m_e\ddot{y}_e + k_ey_e = 0$, with $y_e = \theta_e$, $m_e = (\frac{1}{2}m_A + m_B)R^2$ and $k_e = R^2k$. So, as derived in Eq. (13.12) we have as the solution:

$$\theta_e(t) = A\cos(\omega_n t + \varphi_0) \tag{13.84}$$

$$\omega_n = \sqrt{\frac{k_e}{m_e}} = \sqrt{\frac{k}{\frac{1}{2}m_A + m_B}} \tag{13.85}$$

- Since the vibration starts from rest, we need $\dot{\theta}_e(0) = 0$, from which we find $\varphi_0 = 0$. Then the initial angle is found to be $A = \theta_{e0}$ because $\cos(0) = 1$, obtaining the final solution:

$$\theta_e(t) = \theta_{e0}\cos\left(\sqrt{\frac{k}{\frac{1}{2}m_A + m_B}}t\right) \tag{13.86}$$

13.5 Summary

In this chapter we have analysed vibrations, a periodic type of motion that is important in many fields of science and engineering. Vibrations originate when a position dependent restoring force, like that generated by a spring, acts on a mass. When there are no external forces one obtains free vibrations, while if there are periodic external forces acting, we have forced vibrations. If a velocity dependent force from a damper acts on the mass, the vibrations are damped. There are two main challenges in analysing vibrations: 1. Deriving the correct EoM using the FBD and Newton's or Euler's second law. 2. Solving the EoM, which is mainly the mathematical challenge of solving a second order linear ODE.

- For systems with a restoring force, like mass-spring systems, the EoM as determined using Newton's second law has a special form that causes the motion to exhibit vibrations.
- First determine the static equilibrium position and then determine vibrational motion $y(t)$ with respect to this position.
- Free vibrations
  If there are no time dependent external driving forces, we deal with free vibrations and the following EoM:
  $$m_e\ddot{y}_e + c_e\dot{y}_e + k_e y_e = 0 \qquad (13.87)$$
- If $c_e = 0$, then we are dealing with free *undamped* vibration. In that case the solution of the EoM is $y_e(t) = \Re A e^{i\omega_n t+\varphi_0} = A\cos(\omega_n t + \varphi_0)$, with natural resonance frequency $\omega_n = \sqrt{k_e/m_e}$.
- If $c_e \neq 0$ we are dealing with free *damped* vibration.
  Determine the discriminant $\Delta = c_e^2 - 4m_e k_e$.
  - If $\Delta > 0$ we are dealing with overdamped free vibration with $y_A = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}$ and $\lambda_\pm = \frac{-c_e \pm \sqrt{\Delta}}{2m_e}$.
  - If $\Delta < 0$ we are dealing with underdamped free vibration with $y_A = A e^{-\frac{c_e}{2m_e}t}\cos(\omega_d t + \varphi_0)$ and $\omega_d = \sqrt{\omega_n^2 - c_e^2/(4m_e^2)}$.
  - If $\Delta = 0$ we are dealing with critically damped free vibration with $y_A = A_0 e^{-\frac{c_e}{2m_e}t} + A_t t e^{-\frac{c_e}{2m_e}t}$.
- The system will show forced vibration if the EoM can be written in the form:
  $$m_e\ddot{y}_e + c_e\dot{y}_e + k_e y_e = F_{0,e}\cos(\omega t) \qquad (13.88)$$
- If $c_e = 0$, then we are dealing with forced *undamped* vibration. In that case $\varphi_0 = 0$ such that the steady-state solution of the EoM is $y_e(t) = A\cos(\omega t)$, with $A = F_{0,e}/(k_e - m_e\omega^2)$.

- If $c_e \neq 0$ we are dealing with forced *damped* vibration.
  The steady-state solution of the EoM is:
  $y_{e,f}(t) = A\cos(\omega t + \varphi_0)$.
  The amplitude of the steady-state solution is:
  $A = F_{0,e}/\sqrt{m_e^2(\omega_n^2 - \omega^2)^2 + \omega^2 c_e^2}$.
  The phase difference between force and motion is:
  $\varphi_0 = \arctan[-\omega c/(k - \omega^2 m_A)]$.
- Solutions $y_{e,h}(t)$ of the homogeneous EoM
  $m_e\ddot{y}_{e,h} + c_e\dot{y}_{e,h} + k_e y_{e,h} = 0$ for free vibrations can be added to the steady-state solution $y_{e,f}(t)$ to obtain new solutions of the EoM. For damped vibrations ($c > 0$), the function $y_{e,h}(t)$ reduces to zero after sufficient time, it is therefore called the transient part of the solution. In general, any solution of the forced EoM can always be expressed as $y_e(t) = y_{e,f}(t) + y_{e,h}(t)$.

14 Solution Strategy Dynamics

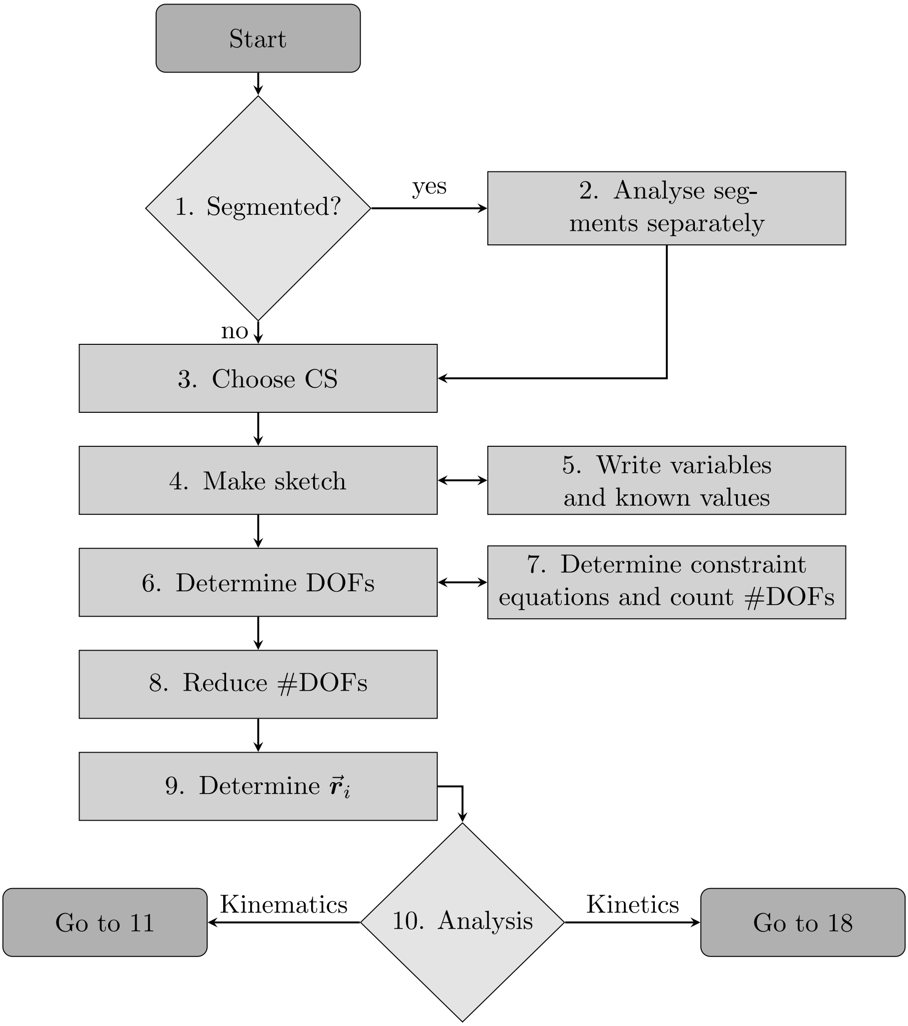
In this textbook we have discussed a wide range of dynamical methods that can be used to solve problems. In order to provide the reader a strategy for solving problem and choosing the right methods we present in this chapter a solution strategy and flow-charts. For each number in the flow-charts a short description is presented. The strategy consists of three flow-charts: preparation, kinematics and kinetics.

14.1 Flowchart preparation

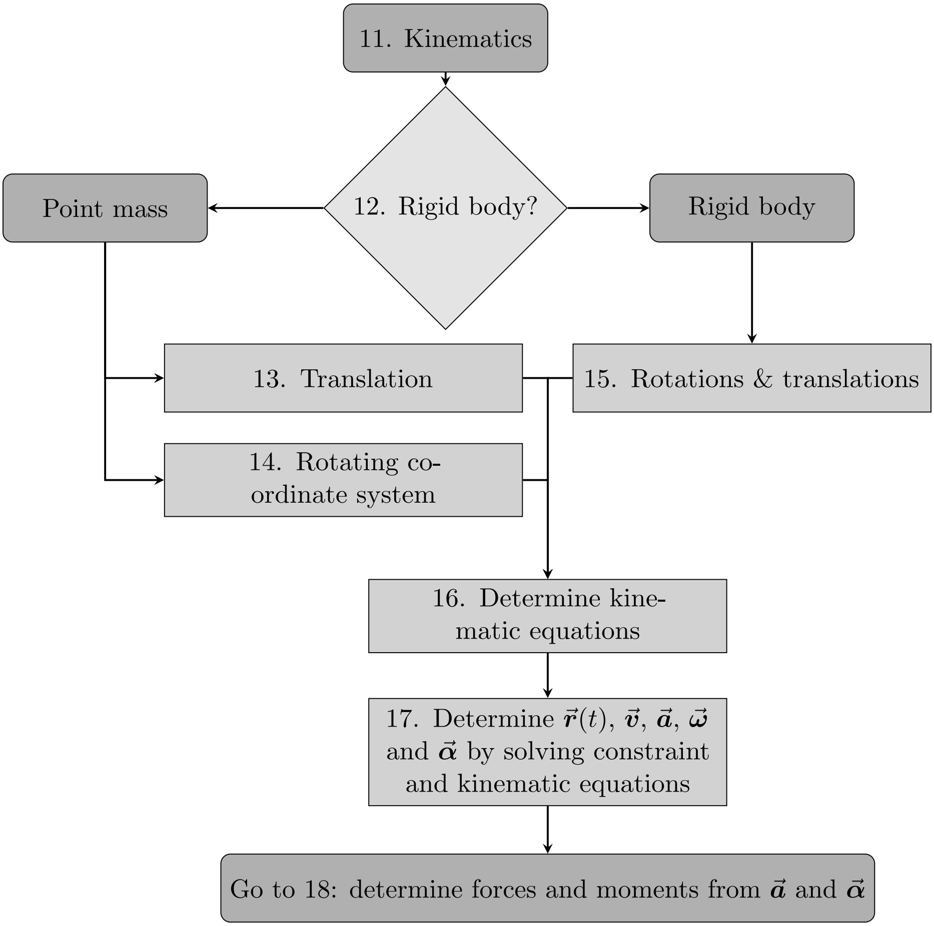
Before starting with the kinematic and kinetic analysis, it is very important to collect all relevant information, simplify it and choose the coordinate systems to analyse the problem along the following steps that are also shown below in the flow-chart preparation:

1. Can the problem be segmented in time?
2. If yes, then split the problem in segments and analyse each segment separately. The initial conditions of a segment (initial positions and velocities) are determined from the end conditions of the preceding segment.
3. Choose one or more coordinate systems (CS) to analyse the kinematics. Use the provided information to choose the system(s) which makes the analysis simplest.
4. Make a sketch of the problem that contains all relevant objects, constraints and coordinate systems.
5. Write down all variables, given values and other relevant equations and information.
6. Determine all degrees of freedom (DOFs) needed to uniquely determine the positions and orientations of all objects. E.g. x_P, y_P for every point mass P and x_G, y_G and ϕ for every rigid body.
7. Determine the constraint equations and count the number of DOFs.
8. Reduce the number of DOF: using constraint equations, e.g. from fixed points, joints, path curves.

9. Determine the position vectors $\vec{\boldsymbol{r}}$ and components of all relevant points using the CS.
10. Determine if a kinematic or kinetic analysis is needed. If there is enough information to solve the problem without considering the effect of forces the problem can be solved with kinematics (go to step 11). Otherwise, first the effect of forces should be determined using kinetics (go to step 18) and then the EoM is solved.



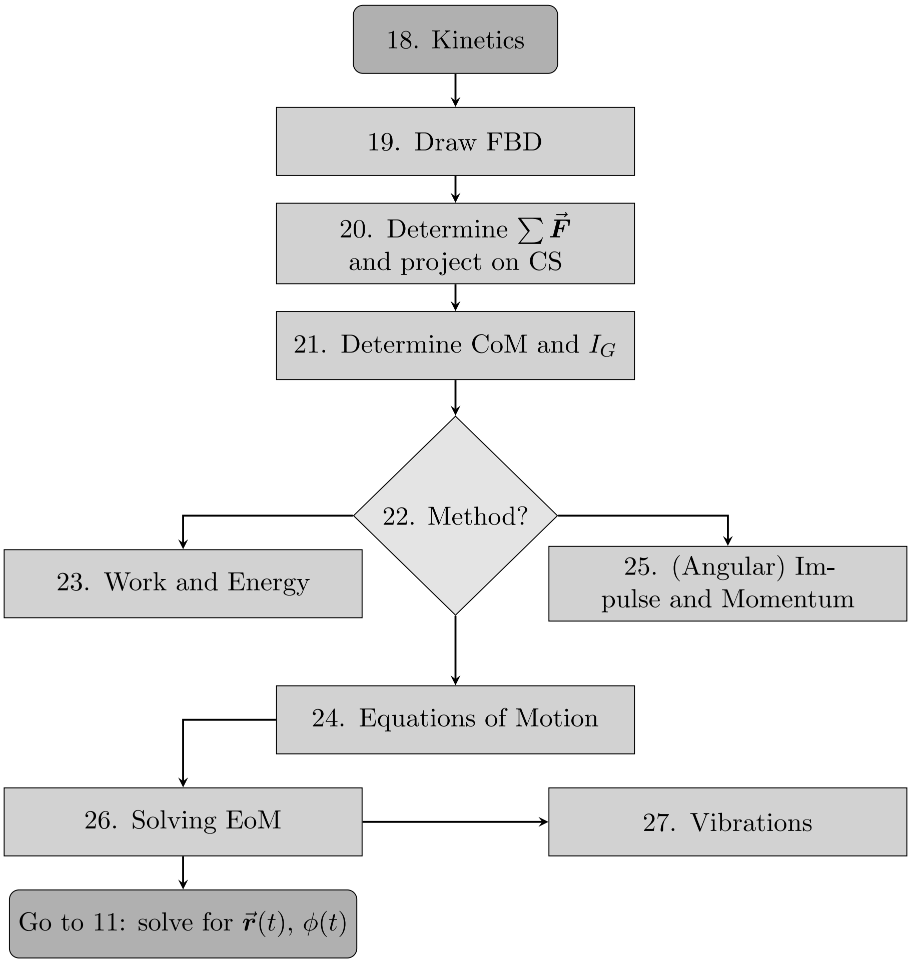
14.2 Flowchart kinematics



11. This flowchart shows the strategy for dealing with kinematics problems.
12. First determine if we deal with a rigid body or point mass, since the kinematics of point masses is significantly simpler.
13. *Kinematics translation:* the goal is to determine position $\vec{\boldsymbol{r}}$, velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ if only one of the three is known. This is done by solving the ODE, e.g. by differentiation and integration over time t, or along the path s or angle ϕ coordinate (see Table 5.1). In 3D we have derived equations to differentiate along Cartesian, natural or cylindrical coordinates (see Sec. 5.9).
14. *Rotating coordinate systems:* the goal is to determine $\vec{\boldsymbol{v}}_B$, $\vec{\boldsymbol{a}}_B$ of point B in an IRF if $\vec{\boldsymbol{v}}'_B$ and $\vec{\boldsymbol{a}}'_B$ are known in a CS that rotates with angular velocity $\vec{\boldsymbol{\Omega}}$, or vice versa (see Sec. 9.7).

15. *Rotating rigid bodies:* the goal is to determine $\vec{\boldsymbol{v}}_B$, $\vec{\boldsymbol{a}}_B$ of a point B that is fixed on a rigid body if $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$ are known or vice versa. Translation of a point is dealt with similar as in step 13. The kinematic equations for rotation of rigid body are discussed in Ch. 9. In particular the instantaneous centre of rotation (IC, see Sec. 9.5.1) can facilitate determination of velocities.
16. The kinematic equations are determined using steps 13-15.
17. Finally the motion $\vec{\boldsymbol{r}}(t)$, $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{a}}$, $\vec{\boldsymbol{\omega}}$, $\vec{\boldsymbol{\alpha}}$ is determined by solving the kinematic and constraint (differential) equations. In case one needs to determine forces: go to step 18 to achieve this with kinetics.

14.3 Flowchart kinetics



18. Kinetics is used to determine the equation of motion from the forces and Newton's and Euler's laws.
19. The first step is to cut objects with mass loose and draw an FBD for each of them (see Sec. 6.8).
20. Use the FBD to determine the sum of forces and sum of moments, and project them on the coordinate system.
21. Determine the centre of mass CoM (see Sec. 7.3.2) and moment of inertia I_G of each object (see Sec. 10.5).
22. The kinetic analysis can be significantly simplified by using the method of work and energy or the method of (angular) impulse and momentum, so carefully choose the right method.
23. *Work and energy* can be used if (see Ch. 7 and Ch. 11)
    - Forces as a function of position $\vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$ or moments as function of angle are known.
    - The trajectory/path is known or does not need to be determined.
    - There is only 1 unknown scalar (e.g. the final speed).

    Conservation of energy can further simplify the analysis if all forces are conservative forces.
24. *(Angular) impulse and momentum* can be used if (see Ch. 8 and Ch. 12):
    - Impulse and/or angular impulse by external forces can be determined.
    - Only changes in (angular) velocity need to be determined, while the positions are approximately constant.
    - In case the sum of external forces and/or moments is zero the analysis can be further simplified using conservation of momentum and/or angular momentum.
    - Besides (angular) momentum conservation, energy conservation or coefficients of restitution can be used to solve e.g. collision problems.
25. *Equations of motion.* Determine the equations of motion for translation and rotation using Euler's laws and project them on the CS to obtain scalar equations (see Ch. 6 and Ch. 10). Combine them with the constraint equations. Note that solving these equations is mostly a mathematical and kinematics challenge.
26. Solve the (differential) equations of motion to obtain the motion $\vec{\boldsymbol{r}}(t)$. Go to step 11 to determine motion of all points using kinematics.
27. *Vibrations* are special EoMs that are solved by special means for driven and free vibrations that can be damped or undamped (see Ch. 13).

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Introductory Dynamics

2D Kinematics and Kinetics of Point Masses and Rigid Bodies

Peter G. Steeneken

Motion is all around us, the universe is full of moving matter and this motion is surprisingly predictable. The field of science and engineering that studies time-dependent motion in the presence of forces is called dynamics. In this book we will introduce the core concepts in dynamics and provide a comprehensive toolset to predict and analyse planar 2D motion of point masses and rigid bodies. The material includes kinematic analysis, Newton's laws, Euler's laws, the equations of motion, work, energy, impulse and momentum. Vector-based methods are discussed for systematically solving essentially any problem in 2D dynamics. The book provides an introduction at bachelor level for science and engineering students, that serves as a basis for more advanced courses in dynamics.

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